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Generalized Hamiltonian Mechanics

A Mathematical Exposition of Non-smooth Dynamical Systems and Classical Hamiltonian Mechanics

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Introduction

Our purpose is to generalize Hamiltonian mechanics to the case in which the energy function (Hamiltonian), H, is a distribution (generalized function) in the sense of SCHWARTZ. We follow the same general program as in the smooth case; see ABRAHAM [1]. Familiarity with the smooth case is helpful, although we have striven to make the exposition self-contained, starting from calculus on manifolds.

The physical motivation for generalized mechanics is clear. Many systems in use actually have singular Hamiltonians. Perhaps the most famous example is that of hard spheres in a box, extensively studied by SINAI. The potential energy in this case is a surface delta function on the walls of the box and on each sphere. Since the usual version of HAMILTON's equations break down (even as distributional equations), we view the flow as a limit of smooth flows. Unfortunately, the variational theorems usually fail in the non-smooth case as is seen from elementary examples; see MARSDEN [5].

One of the main theorems concerns existence of a flow (defined almost everywhere) for a Hamiltonian whose singular support has measure zero. As the flow need not be unique, we regard a physical system as being specified by H and a sequence of (suitably well behaved) smooth functions H_i converging to H. This will then fix the flow. The usual theorems on conservation of energy, Poisson brackets and Liouville's theorem then carry over, although the methods are quite different from the smooth case.

Chapter one deals with global distribution theory on manifolds. A certain amount of this material is found in DE RHAM [1], although our approach is slightly different. The main new concepts are the generalized Lie derivative and generalized vectorfield with its associated flow.

Chapter two studies Hamiltonian systems in particular. Conservation laws are given explicitly and the connection with the Bohr-Dirac "correspondance principle" is proven. (This seems to have been first stated for manifolds by SEGAL [1, p. 475].) We also give some applications to statistical mechanics (a global virial theorem) and show that the generalized eigenfunctions of a smooth Hamiltonian system uniquely determine the flow. In the appendix to section 10, we show how our methods yield non-smooth geodesic flows.

The Lagrangian formalism is not discussed. This offers no difficulty in practice, as it may be converted formally to a Hamiltonian one (ABRAHAM [1, § 18]).

Of course, not all the theorems here are claimed to be new. We have striven for clarity of the exposition rather than a concise report of new results. For the reader interested in a global Hamiltonian formulation of some classical continuum systems and quantum mechanics, we refer to MARSDEN [1].

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Glossary of Symbols

Our notation follows that of ABRAHAM [1] almost exclusively. However, the following brief summary may be helpful. Numbers in brackets refer to the following sections where the definitions may be found and the prefix "A" refers to ABRAHAM [1].

$\boldsymbol{R}$, resp. $\boldsymbol{C}$	the reals, resp. complex numbers,
$\boldsymbol{R}^n$	Euclidean n-space, $\boldsymbol{R}\times\boldsymbol{R}\times\cdots\times\boldsymbol{R}$,
M	finite dimensional smooth, orientable manifold (A 3.2, A 11.6),
$f: M\to N$	mapping,
$m\mapsto f(m)$	effect of the mapping f,
$\{g_i\}$	partition of unity (A 11.2),
$T_m M$	tangent space to M at $m\in M$ (A 5.3),
TM	tangent bundle, $\bigcup\{T_m M: m\in M\}$ (A 5.3),
$T^* M$	cotangent bundle (A 6.14),
$\mathscr{F}(M)$	smooth (C^∞) maps $f: M\to \boldsymbol{R}$ (A 6.15),
$\mathscr{F}_c(M)$	elements of $\mathscr{F}(M)$ with compact support (1.2, 2.1),
$\mathscr{F}_c(M)^*$	linear maps $\alpha: \mathscr{F}_c(M)\to\boldsymbol{R}$ (1.1 *ff.*),

$\mathscr{F}(M)'$	generalized functions (distributions) (1.2, 2.1),
δ_m	delta function at $m \in M$ (22 ff.),
$\mathfrak{X}(M)$	C^∞ vectorfields; sections of TM (A 6.15),
$\mathfrak{X}^*(M)$	C^∞ one-forms; sections of T^*M (A 6.15),
$\mathfrak{X}(M)'$	generalized vectorfields (4.3),
$\Omega^k(M)$	k-forms (A 10.3),
$\Omega^k(M)'$	generalized k-forms $\equiv$ courants (2.1),
$\mathscr{T}_s^r(M)$	tensorfields of type (r, s) (A 6.15),
$\varphi: \Omega^k(M) \to \Omega^k(M)'$	the natural embedding (2.1),
$\mathscr{T}_s^r(M)'$	generalized tensorfields of type (r, s) (4.3),
$\alpha \wedge \beta$	exterior product (A 10.2), (3.1),
d	exterior derivative (A 10.5), (3.2),
$[X, Y]$	Lie bracket (A 10.12), (5.3),
L_X	Lie derivative (A 8.18), (3.3, 5.5),
i_X	inner product (A 10.12), (3.4, 5.4),
Ω	volume on M (A 11.4),
$\det_\Omega f$	determinant (Jacobian) of $f: M \to M$ (A 11.18),
$\operatorname{div}_\Omega X$	divergence of $X \in \mathfrak{X}(M)'$ (A 11.22), (5.6),
μ_Ω	measure determined by Ω (A 12.9),
$F_*: \Omega^k(N) \to \Omega^k(M)$	pull back by $F: M \to N$ (A 10.7), (2.5),
F_t	flow; $F_t(m) = F(t, m)$, $(t, m) \in R \times M$ (A 7.5), (6.1),
ω	symplectic form (A 14.8),
$\{f, g\}$	Poisson bracket (A 14.23), (7.4),
X_H	Hamiltonian vector field of $H \in \mathscr{F}(M)'$ (A 14.23), (7.3),
$\Phi_g: M \to M$	action of a Lie group (A 22.8),
P_X	momentum of X; $P_X(\alpha_m) = \alpha_m \cdot X(m)$ (9.1),
$A \backslash B$	set-theoretic difference,
A^c	closure of A (A p. 236),
A^{sc}	sequential closure of A (1.1),
$\operatorname{bd} A$	boundary of A (A p. 236),
$(V, V_0, \operatorname{bd} V_0)$	compact orientable manifold with boundary (A 12.11).

Chapter One: Distributions on Manifolds

§1. Generalized Quantities

From the global point of view, the most natural approach to distribution theory is to regard a distributional "quantity" as a weak limit of "quantities", thus avoiding unnecessary coordinates. The general situation, which will recur many times in the sequel is as follows:

1.1. Definition. *Let T be a (real) topological vector space, S a vector space and*

$$\varphi: S \to T$$

a monomorphism (one-to-one linear map). Elements of $\{\varphi(S)\}^{sc}$, which denotes all countable limits of elements of $\varphi(S)$ (sequential closure) are called **generalized S quantities** *with respect to φ and T, and are denoted S'. We often identify S and $\varphi(S)$.*

In this definition it is essential to use the sequential closure rather than the closure to recover the Schwartz theory. The reason is as follows. In 1.1, the space T is often L^*, all real linear maps $\alpha: L \to R$ for a vector space L. On L^* we put the relative product topology, sometimes called the weak $*$-topology or the pointwise convergence topology. It is determined by: for any net $\alpha_i \in L^*$, $\alpha_i \to \alpha \in L^*$ iff $\alpha_i(v) \to \alpha(v) \in R$ for all $v \in L$. It can be shown that any point separating subspace of L^* is dense in L^*. See KELLEY [1, p. 108–109]. (It is easy to see that L^* is a topological vector space.)

This situation occurs in the main theorem of the local theory which we shall employ in a few proofs and is as follows:

1.2. Theorem (SCHWARTZ). *Let $\mathscr{F}_c(R^n)$ denote the smooth real functions on R^n with compact support and define*

$$\varphi: \mathscr{F}(R^n) \to \mathscr{F}_c(R^n)^*$$

$$\varphi(f) \cdot g = \int f g \, d\mu$$

where μ is Lebesgue measure. Then $\alpha \in \mathscr{F}_c(R^n)^*$ is a generalized $\mathscr{F}(R^n)$ quantity iff for every sequence $f_n \in \mathscr{F}_c(R^n)$ with supports in a compact set K and f_n together with all its derivatives converge uniformly to zero, then $\alpha(f_n) \to 0$.

Furthermore, $\mathscr{F}(R^n)'$ is sequentially closed and sequential convergence defines a topology (not obvious!).

For the proof we refer to any standard text on distribution theory (SCHWARTZ [3], GELFAND [1], GARSOUX [1], or YOSIDA [1]).

As a further illustration, which will not be used in the sequel, we consider the following example, often referred to as **vector valued distributions**, not to be confused with generalized vectorfields, which we shall study later.

1.3. Definition. *Let A be an orientable manifold and $\pi: V \to A$ be a vector bundle over A equipped with a Riemannian metric $\langle\ ,\ \rangle$ (a metric on the fibers of π). Let $\Gamma_c(\pi)$ denote the smooth sections of π with compact support. Fix a volume Ω on A, and let μ be the corresponding measure.*

Define

$$\varphi: \Gamma(\pi) \to \Gamma_c(\pi)^*$$

by

$$\varphi(f) \cdot g = \int_A \langle f(a), g(a) \rangle \, d\mu(a).$$

Then generalized $\Gamma(\pi)$ quantities are called **generalized sections** *of π (using the weak $*$-topology).*

Here it is easy to verify that φ is a monomorphism. Notice that π need not have finite dimensional fibers, although A is finite dimensional, so this includes **operator valued distributions** as well (Hilbert-Schmidt operators, for example).

§ 2. Generalized Forms

This section develops some of the basic properties of generalized forms such as the action of orientation preserving diffeomorphisms, and integration. Integration is always done with respect to a fixed orientation (ABRAHAM [1, § 12]).

(use usual $\langle\ ,\ \rangle$ for forms and any volume ... are isomorphic to gen. forms. Similarly for generalized vectorfields.)

2.1. Definition. *Let* $\dim M = n$ *and* $\Omega_c^k(M)$ *denote the k-forms on M which are smooth and have compact support. Define*

$$\varphi\colon \Omega^k(M) \to \Omega_c^{n-k}(M)^*$$

by

$$\varphi(\alpha)\cdot\beta = \int \alpha\wedge\beta .$$

Then generalized $\Omega^k(M)$ *quantities are called* **generalized** k**-forms** *and are denoted* $\Omega^k(M)'$. *We write* $\mathscr{F}_c(M) = \Omega_c^0(M)$ *and* $\mathscr{F}(M)' = \Omega^0(M)'$.

To justify the definition we must check that φ is a monomorphism. It is obviously R-linear. Suppose $\alpha(m) \neq 0$. Using a local chart, we can find $\beta\in\Omega_c^{n-k}(M)$ so that in this chart $\alpha\wedge\beta = f\,dx^1\wedge\cdots\wedge dx^n$ where $f \geqq 0, f(m) > 0$ and has compact support. Then $\int\alpha\wedge\beta \neq 0$, which proves the assertion.

Thus we regard $\Omega^k(M)\subset\Omega^k(M)'$. Similarly we may regard $\Omega_{LI}^k(M)\subset\Omega^k(M)'$, where $\Omega_{LI}^k(M)$ denotes the (equivalence classes of) locally integrable k-forms on M. (That is, for each $\beta\in\Omega_c^{n-k}(M)$ and volume Ω, $\alpha\wedge\beta = f_\beta\,\Omega$ where f_β is locally integrable with respect to μ_Ω.) (This condition depends only on the orientation.) To see that $\varphi\colon \Omega_{LI}^k(M)\to\Omega^k(M)'$ is a monomorphism, suppose $\int f_\beta\,d\mu = 0$ for all β. For $U\subset M$ open with U^c compact there are $f_n\in\mathscr{F}_c(M)$ so $f_n\uparrow C_U$, the characteristic function of U. Hence, by a theorem of Lebesgue,

$$\int_U f_\beta\,d\mu = \lim_{n\to\infty}\int f_\beta f_n\,d\mu = \lim_{n\to\infty}\int f_{f_n\beta}\,d\mu = 0 .$$

Thus, as the open sets generate the measurable ones, $f_\beta = 0$ almost everywhere. Hence $\alpha = 0$. To see that $\varphi(\alpha)\in\Omega^k(M)'$, we may use standard approximation theorems, for example BERBERIAN [1, p. 220] with his $\mathscr{L}$ replaced by $\mathscr{F}_c$, or we can use 1.2 locally (see 3.7 below for coordinate language).

The next theorem gives a connection between convergence as distributions and pointwise convergence. Recall that a family $\{f_\alpha\}$ of real continuous functions is called **equi-continuous** at $m\in M$ iff for all $\varepsilon > 0$ there is a neighborhood U of m such that $m'\in U$ implies $|f_\alpha(m) - f_\alpha(m')| < \varepsilon$ for all α.

2.2. Proposition. *Suppose* f_n *and* f *are continuous at* $x\in M$ *and the family* $\{f_n\}$ *is equi-continuous at* x. *Then if* $f_n\to f$ *in* $\mathscr{F}(M)'$, $f_n(x)\to f(x)$.

Proof. Consider those n such that $f_n(x) > f(x)$. We shall show this subsequence converges to $f(x)$. The case $f_n(x) \leqq f(x)$ is similar. Given $\varepsilon > 0$, choose a neighborhood U of x such that $y\in U$ implies $|f_n(x) - f_n(y)| < \varepsilon/3$, $|f(x) - f(y)| < \varepsilon/3$ and $f_n(y) > f(y)$, for all n.

Suppose not ∃ subseq f_n bigger by ε away from f at x

Con suppose $f_n > f$.

Now there is an N so that $n \geqq N$ implies there is a $y\in U$ so $|f_n(y) - f(y)| < \varepsilon/3$. For, if not, choose a smooth map $\varphi\colon M\to R$; $0\leqq\varphi\leqq 1$ so $\varphi = 0$ outside U and $\varphi = 1$ on a neighborhood $V\subset U$. Then

$$\int (f_n - f)\,\varphi\,d\mu \geqq \varepsilon\,\mu(V)/3$$

contradicting $f_n\to f$ in $\mathscr{F}(M)'$.

Thus $n\geqq N$ implies

$$|f_n(x) - f(x)| \leqq |f_n(x) - f_n(y)| + |f_n(y) - f(y)| + |f(y) - f(x)| < \varepsilon . \quad \square$$

Without equi-continuity, the proposition is false. For example, $\sin(nx)\to 0$ in $\mathscr{F}(R)'$ but not pointwise. For further results along these lines see BELTRAMI [1].

Examples of generalized forms are commonplace. For instance, if $m\in M$ define $\delta_m\in\Omega^n(M)'$ by $\delta_m(f)=f(m)$, the Dirac delta function. Notice that it is properly interpreted as a generalized n-form, or measure and *not* a generalized function. That $\delta_m\in\Omega^n(M)'$ is easy to see. In fact suppose U_n are open sets, $U_n\downarrow\{m\}$ and f_n has support in U_n and $\int f_n\,\Omega=1$. Then $f_n\Omega\to\delta_m$.

Similarly we can define surface δ-functions. Let $S\subset M$ be an orientable submanifold of codimension k. Define $\delta_S\in\Omega^k(M)'$ by

$$\delta_S(\alpha)=\int_S i_*\alpha;\qquad \alpha\in\Omega_c^{n-k}(M)$$

where $i\colon S\to M$ is inclusion and $*$ denotes the pull-back. It is an easy exercise in approximation (or by 1.2) to show that these are honest generalized forms.

2.3. Definitions. *A generalized n-form α is called* **positive** *iff $f\in\mathscr{F}_c(M)$, $f\geqq 0$ implies $\alpha(f)\geqq 0$. Similarly, $g\in\mathscr{F}(M)'$ is called* **positive** *relative to the orientation iff $h\in\mathscr{F}_c(M)$, $h\geqq 0$ implies $g(h\Omega)\geqq 0$ where Ω is a representative of the orientation.*

For $\alpha\in\Omega_c^{n-k}(M)^*$, $U\subset M$ open, we define $\alpha\,|\,U\in\Omega_c^{n-k}(U)^*$ by $\alpha\,|\,U(\beta)=\alpha(\beta)$, and

$$\operatorname{supp}\alpha=M\backslash\bigcup\{U\subset M\colon\ U\ open,\ \alpha\,|\,U=0\}$$

the **support** *of α, and*

$$\operatorname{sing\,supp}\alpha=M\backslash\bigcup\{U\subset M\colon\ U\ open,\ \alpha\,|\,U\ \text{ is smooth}\}$$

the **singular support**, *where "$\alpha\,|\,U$ smooth" means $\alpha\,|\,U\in\Omega^k(M)\subset\Omega_c^{n-k}(M)^*$.*

It is easy to check that these definitions coincide with the usual ones. For example, if $f\colon M\to R$ is locally integrable, then $f\geqq 0$, a.e., iff $\varphi(f)\geqq 0$ iff $\varphi(f\Omega)\geqq 0$. Using a partition of unity, we conclude at once that if $\alpha\,|\,U_i=0$ for an open cover $\{U_i\}$, then $\alpha=0$. From this fact it is easy to see that the definition of support coincides with the usual one if α is smooth or locally integrable.

Also, if $\alpha\in\Omega_c^{n-k}(M)^*$ and for some open cover $\{U_i\}$, $\alpha\,|\,U_i\in\Omega^k(U_i)'$, then $\alpha\in\Omega^k(M)'$. In fact, if $\{g_i\}$ is a subordinate partition of unity and $\varphi(\alpha_n^i)\to\alpha\,|\,U_i$ in $\Omega^k(U_i)$, then $\varphi(\beta_j)\to\alpha$ where

$$\beta_j=\sum_i g_i\,\alpha_j^i.$$

2.4. Theorem (RIESZ-MARKOFF-GELFAND). *Let ω be a positive generalized n-form on M. Then there is a unique regular Borel measure μ_ω on M such that*

$$\omega(f)=\int f\,d\mu_\omega$$

for each $f\in\mathscr{F}_c(M)$ (regular Borel$\equiv$positive Rodon measure).

Proof. For g continuous with compact support, $g\geqq 0$, define $\omega(g)=\sup\{\omega(f)\colon f\in\mathscr{F}_c(M)$ and $0\leqq f\leqq g\}$. Then it is an easy exercise to check that this extension maps into R and is linear and positive. Hence it is represented by a measure (a Radon measure). $\square$

This proposition seems to have been first observed by GELFAND-VILENKIN [4]. The converse is also true; that is, each Borel measure (in fact every Radon measure, positive or not) defines a $\omega \in \Omega^{n\prime}$ by the above formula.

For example, the measure associated with δ_m is the point measure at m. In general a non-positive generalized n-form is not associated with a signed measure. For example, on R consider $\delta'(f) = -f'(0)$.

Note that from the Radon-Nikodym theorem, if ω is a positive smooth n-form, and $\alpha \in \Omega^n(M)'$, $\alpha \geqq 0$, then $\mu_\alpha \ll \mu_\omega$ (absolute continuity) iff $\alpha = f\omega$ for some f locally integrable.

The action of diffeomorphisms on forms extends uniquely to generalized ones as follows:

2.5. Theorem. *Let F: $M \to N$ be an orientation preserving diffeomorphism. Then F^*: $\Omega^k(M) \to \Omega^k(N)$ has a unique extension as a continuous map F^*: $\Omega^k(M)' \to \Omega^k(N)'$. In fact,*

$$(F^* \alpha)(\beta) = \alpha(F_* \beta).$$

Also F^* is an isomorphism and homeomorphism and satisfies

$$(F \circ G)^* = F^* \circ G^*.$$

{See deRham p55 if F only a smooth map}

Proof. Consider F so defined. It is obviously continuous. To show that it coincides with the usual definition we must show, for $\alpha \in \Omega^k(M)$, $\beta \in \Omega_c^{n-k}(M)$,

$$\int \alpha \wedge F_* \beta = \int F^* \alpha \wedge \beta$$

which is clear by the change of variables formula (ABRAHAM [1, 12.7]). It is also clear that $\alpha \in \Omega^k(M)'$ implies $F^* \alpha \in \Omega^k(N)'$. The rest is obvious. $\square$

Finally, in this séction we discuss briefly the integration of generalized forms.

2.6. Definition. *Let ω be a generalized n-form with compact support. Let $\{g_i\}$ be a partition of unity with* supp(g_i) *compact. Define*

$$\int \omega = \sum_i \omega(g_i) \in R$$

(the sum has only a finite number of non-zero terms).

Clearly, the definition is independent of the partition of unity, since

$$\sum_i \omega(g_i) = \sum_{i,j} \omega(g_i h_j) = \sum_i \omega(h_j)$$

and coincides with the usual integral when ω is smooth. If ω does not have compact support but the sum in 2.6 converges (possibly to $+\infty$) independent of $\{g_i\}$, we say it is **integrable**.

Positive generalized n-forms are integrable and

$$\int \omega = \sum_i \omega(g_i) \leqq \infty$$

and this coincides with $\int d\mu_\omega$ by Lebesgue's monotone convergence theorem.

Also, if Ω is a volume and f is μ_Ω-integrable, then $\int f\Omega = \int f\, d\mu_\Omega$ if we use the positive and negative parts of f.

2.7. Theorem (Change of Variables). *Let $F: M \to M$ be an orientation preserving diffeomorphism. Recall that if Ω is a volume, $\det F \in \mathscr{F}(M)$ is defined by $F_* \Omega = (\det F)\Omega$. Then*

(i) *if $\omega \in \Omega^n(N)'$ has compact support, or is positive or is integrable, then $\int F^* \omega = \int \omega$;*

(ii) *if μ is the measure of Ω and f is μ-integrable, then*

$$\int f\, d\mu = \int (f \circ F)(\det F)\, d\mu .$$

Proof. The first part is clear since if $\{g_i\}$ is a partition of unity, so is $\{g_i \circ F\}$. For the second part it is sufficient to show that $F_* \mu = (\det F)\mu$ by HALMOS [1, p. 163]. Thus, we must show $F_* \mu_\Omega = \mu_{F_* \Omega}$. But this follows from

$$\int g\, F_* \Omega = \int (F^* g)\, d\mu_\Omega = \int g\, d(F_* \mu_\Omega). \quad \square$$

The integral $\int : \Omega^n(M)'_c \to R$ is not continuous. For example, on R, $\delta_{m_i} \to 0$ if $m_i \to \infty$, but $\int \delta_{m_i} = 1$. Nevertheless, we have:

2.8. Proposition. *$\int$ is the unique mapping from generalized n-forms with compact support to R such that $\int$ is the usual integral on $\Omega^n_c(M)$ and if $\omega_i \to \omega$ all having supports in some compact set, then $\int \omega_i \to \int \omega$.*

Proof. First, $\int$ has this property, for if A is the compact set and $\{g_i\}$ is any partition of unity,

$$\sum_i \omega(g_i) = \sum_i \operatorname*{limit}_j \omega_j(g_i) = \operatorname*{limit}_j \sum_i \omega_j(g_i)$$

since $\sum_i$ is a fixed finite sum.

For uniqueness, if $\omega \in \Omega^n(M)'$ and has compact support, there are $\omega_i \in \Omega^n_c(M)$ so $\omega_i \to \omega$ and all have support in some compact set, as we see by multiplying by a suitable function with compact support. $\square$

Finally we remark that the hypothesis that diffeomorphisms preserve orientation, used in this section, is really not restrictive for connected manifolds, for if F is orientation reversing for Ω, then it is orientation preserving for $-\Omega$.

§ 3. Exterior Algebra

This section covers more analytical aspects of generalized forms. The main goals are to extend the exterior and Lie derivatives to the generalized case. We state the de Rham regularization theorem and deduce a few important consequences such as the generalized Poincaré lemma. We also prove the important "flow theorem" relating the Lie derivative to its flow in the generalized case.

3.1. Definition. *Let $\alpha \in \Omega^k(M)'$ and $\beta \in \Omega^l(M)$. Then define $\alpha \wedge \beta \in \Omega^{k+l}(M)'$ by*

$$(\alpha \wedge \beta)(\gamma) = \alpha(\beta \wedge \gamma)$$

and put $\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta$.

Clearly this coincides with the usual definition and is uniquely determined by continuity in α.

The basic theorem on the exterior derivative is:

3.2. Theorem. *The exterior derivative* d: $\Omega^k(M) \to \Omega^{k+1}(M)$ *has a unique extension to a continuous map (denoted by the same letter)*

$$d\colon\ \Omega^k(M)' \to \Omega^{k+1}(M)'.$$

In fact, $d\alpha(\beta) = (-1)^{k+1}\alpha(d\beta)$ *and satisfies*

(i) d *is* $\mathbf{R}$*-linear and,* $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$ *for* $\alpha \in \Omega^k(M)'$, $\beta \in \Omega^l(M)$,

(ii) $d \circ d = 0$,

(iii) *if* $\alpha \in \Omega^{n-1}(M)'$ *and has compact support,* $\int d\alpha = 0$.

Proof. We first prove (iii) in case α is smooth. Let $\{g_i\}$ be a partition of unity subordinate to some atlas, so that

$$\int d\alpha = \sum_i \int g_i\, d\alpha = \sum_i \int d(g_i \alpha)$$

since

$$\sum_i g_i = 1.$$

Hence it is sufficient to prove the result in $\mathbf{R}^n$. But there it is obvious by Stokes' theorem (ABRAHAM [1, p. 82]).

Now d so defined is clearly continuous. To show that the result coincides with the usual one, note that

$$\int (d\alpha) \wedge \beta = \int d(\alpha \wedge \beta) + (-1)^{k+1} \int \alpha \wedge d\beta$$
$$= (-1)^{k+1} \int \alpha \wedge d\beta,$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega_c^{n-k-1}(M)$.

Now (i) and (ii) are clear by continuity or directly. For (iii),

$$\int d\alpha = \sum_i d\alpha(g_i) = \sum_i (-1)^n \alpha(d g_i) = 0$$

(the sums are finite). $\square$

Notice that 3.2 (iii) gives an easy proof that a compact orientable n-manifold has n^{th} de Rham cohomology group non-trivial (same for Cech cohomology by the de Rham isomorphism). In fact, if Ω is a volume $d\Omega = 0$; but if $\Omega = d\alpha$ we would conclude $\int d\mu_\Omega = 0$.

If F: $M \to N$ is an orientation preserving diffeomorphism, then we conclude by continuity, or directly that

$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$$

and

$$F^* d\alpha = dF^*\alpha$$

for $\alpha \in \Omega^k(M)'$ and $\beta \in \Omega^l(M)$.

3.3. Theorem. *Let X be vectorfield (C^∞) on M and L_X: $\Omega^k(M)\to\Omega^k(M)$ the Lie derivative. Then L_X has a unique continuous extension (denoted by the same letter)*

$$L_X\colon\ \Omega^k(M)'\to\Omega^k(M)'.$$

In fact,

$$(L_X\alpha)\cdot\beta=-\alpha(L_X\beta).$$

Moreover, we have

(i) *L_X is $\mathbf{R}$-linear, and*

$$L_X(\alpha\wedge\beta)=L_X\alpha\wedge\beta+\alpha\wedge L_X\beta$$

for $\alpha\in\Omega^k(M)'$, $\beta\in\Omega^l(M)$;

(ii) $L_X\,d\alpha=d\,L_X\,\alpha$;

(iii) *if F: $M\to N$ is an orientation preserving diffeomorphism, $F^*(L_X\alpha)=L_{F^*X}F^*\alpha$;*

(iv) *if β has compact support,*

$$\int L_X\alpha\wedge\beta=-\int\alpha\wedge L_X\beta.$$

Proof. Since $L_X(\alpha\wedge\beta)=d\,i_X(\alpha\wedge\beta)$ if α,β are smooth, (iv) follows in this case from 3.2 (iii), and the rest of the proof proceeds like the proof of 3.2. $\square$

In a similar way we have the following:

3.4. Theorem. *Let X be a (smooth) vectorfield on M and i_X: $\Omega^k(M)\to\Omega^{k-1}(M)$ the inner product. Then i_X has a unique continuous extension*

$$i_X\colon\ \Omega^k(M)'\to\Omega^{k-1}(M)'.$$

In fact $(i_X\alpha)\cdot\beta=(-1)^{k+1}\,\alpha(i_X\,\beta)$ for $\alpha\in\Omega^k(M)'$. Moreover,

(i) *i_X is $\mathbf{R}$-linear, and $i_X(\alpha\wedge\beta)=(i_X\alpha)\wedge\beta+(-1)^k\,\alpha\wedge i_X\,\beta$;*

(ii) $i_X\circ i_X=0$;

(iii) $L_X=i_X\circ d+d\circ i_X$;

(iv) $i_{[X,Y]}=L_X\circ i_Y-i_Y\circ L_X$;

(v) $L_{fX}=f L_X+(df)\wedge i_X$;

(vi) $L_{[X,Y]}=L_X\circ L_Y-L_Y\circ L_X$;

(vii) *if F: $M\to N$ is an orientation preserving diffeomorphism, then $F^*(i_X\alpha)=i_{F^*X}F^*\alpha$.*

The next theorem is stated without proof and may be found in DE RHAM [1, p. 80]. (This theorem plays an important role in DE RHAM's study of the homology of manifolds and says that the cohomology and generalized cohomology groups are equal.)

A subset B of $\Omega_c^k(M)$ is called **bounded** iff for any $p\geqq 0$ and some covering by coordinate charts, all derivatives up to order p are uniformly bounded and all elements of B have their support in some compact set.

3.5. Theorem (SOBOLEV-SCHWARTZ-DE RHAM regularization). *There exists a sequence of R-linear maps*

$$R_n\colon \Omega^k(M)' \to \Omega^k(M)$$

$$A_n\colon \Omega^k(M)' \to \Omega^{k-1}(M)'$$

such that for each $\alpha \in \Omega^k(M)'$,

$$R_n\alpha - \alpha = dA_n\alpha + A_n\, d\alpha$$

and $A_n(\Omega^k(M)) \subset \Omega^{k-1}(M)$; *if* $f \in \mathscr{F}(M)'$, $A_n f = 0$. *Also,* $R_n\,\alpha \to \alpha$ *and* $A_n\alpha \to \alpha$ *uniformly on bounded sets.*

Of interest in mechanics is a consequence, the *generalized Poincaré lemma* (3.6). Recall that α is **closed** iff $d\alpha = 0$ and is **exact** iff $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)'$. Clearly if α is exact, then α is closed, and α is closed iff for all exact $\gamma \in \Omega_c^{n-k}(M)$, $\alpha(\gamma) = 0$.

3.6. Corollary. (i) *Suppose* $\alpha \in \Omega^k(M)'$ *and* $d\alpha = 0$. *Then for each* $m \in M$, *there is a neighborhood U of m and a generalized* $k-1$ *form* γ *on M so that*

$$\alpha\,|\,U = d\gamma\,|\,U, \qquad k \geqq 1;$$

(ii) *if* $f \in \mathscr{F}(M)'$ *and* $df = 0$, *and M is connected, then f is constant.*

Proof. (i) $\alpha = R_n\alpha - dA_n\alpha$ so that $dR_n\alpha = 0$. Hence by the smooth Poincaré lemma, $R_n\alpha$ is locally exact. For (ii) we have $f = R_n f$, so f is smooth and hence constant. □

For a direct alternative proof of (i), see MARSDEN [3].

Another theorem of basic importance in mechanics is the following:

3.7. Theorem (Flow Theorem). *Suppose X is a smooth vectorfield on M with (complete) flow* F_t. *Then for each* $\alpha \in \Omega^k(M)'$, *the map*

$$t \mapsto F_{t*}\alpha \in \Omega^k(M)'$$

is differentiable and

$$F_{\tau*}L_X\alpha = \frac{d}{dt}(F_{t*}\alpha) \qquad \text{at } t = \tau.$$

In particular, $L_X\alpha = 0$ *iff* $\alpha = F_{t*}\alpha$ *for all t.*

Proof. Let R_n denote the smoothing operator of 3.5 and fix $a > 0$, and $\beta \in \Omega_c^{n-k}$. Let $g_n(t) = F_{t*}(R_n\alpha)\cdot\beta$ and $f(t) = F_{t*}(\alpha)\cdot\beta$. Now for $-a \leqq t \leqq a$ the set of all $F_t^*\beta$ is clearly a bounded set as F_t is a smooth map on $R \times M$. Therefore, g_n converges uniformly to f for $-a \leqq t \leqq a$. However, the derivative of g_n, say $g_n' = L_X(F_{t*}(R_n\alpha))\cdot\beta$, converges uniformly to $L_X(F_{t*}\alpha)\cdot\beta$ by the smooth flow theorem and the same boundedness arguement. Therefore by an elementary theorem in analysis (APOSTOL [1, p. 402]) the derivative of f exists on $-a \leqq t \leqq a$ and equals $L_X(F_{t*}\alpha)\cdot\beta$. Since a was arbitrary, we have the result. □

For a direct alternative proof using 1.2 in local coordinates, see MARSDEN [3].

This theorem is the analogue of ABRAHAM [1, 8.20] in the smooth case.

Thus, if X is a smooth vectorfield, it induces a one-parameter group (flow) on $\mathscr{F}(M)'$ and $\Omega^k(M)'$ in a natural way with infinitesimal generator L_X. We can recover the original flow according to the following:

3.8. Proposition. *Let X be a smooth vectorfield on M with (complete) flow F_t. If $m_t = F_t(m)$, then*

$$F_t^* \delta_m = \delta_{m_t}.$$

Also, $m \in M$ is a critical point of X (i.e., $X(m)=0$), iff

$$F_t^* \delta_m = \delta_m \quad \text{iff} \quad L_X \delta_m = 0.$$

The proof is immediate from 3.7 and the definitions.

Finally, in this section we briefly describe the more familiar coordinate language, leaving proofs to the reader.

For $f \in \mathscr{F}(M)'$, we define, in a coordinate chart, $\partial f/\partial x^i \in \mathscr{F}(U)'$ by

$$\frac{\partial f}{\partial x^i}(g\, dx^1 \wedge \cdots \wedge dx^n) = -f\left(\frac{\partial g}{\partial x^i}\, dx^1 \wedge \cdots \wedge dx^n\right).$$

Then we see that this equals

$$(-1)^{i+1}\, df(g\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n)$$

and so coincides with the usual derivative when f is smooth and

$$df = \sum \frac{\partial f}{\partial x^i} \wedge dx^i.$$

For $\omega \in \Omega^n(M)'$ and Ω a volume there is a unique generalized function f, so $\omega = f\Omega$.

Finally, every $\alpha \in \Omega^k(U)'$ can be written uniquely

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for $\alpha_{i_1 \ldots i_k}$ generalized functions.

§ 4. Generalized Tensors

Although the only generalized tensors required later are forms and vectorfields, we briefly consider the general case for completeness.

We begin with an alternative description of generalized forms which leads naturally to the definition of generalized tensor. Recall that there is an isomorphism

$$\psi\colon\ \Omega^k(M) \to L^k_a$$

where L^k_a denotes the alternating $\mathscr{F}(M)$-multilinear maps $\alpha\colon \mathscr{X}(M) \times \cdots \times \mathscr{X}(M) \to \mathscr{F}(M)$. (ABRAHAM [1, § 8].) In fact,

$$\psi(\alpha)\cdot(X_1, \ldots, X_k)(m) = \alpha(m)\cdot\big(X_1(m), \ldots, X_k(m)\big).$$

4.1. Proposition. *ψ has a unique continuous extension*

$$\psi\colon\ \Omega^k(M)' \to L^{k'}_a$$

where $L_a^{k'}$ is the alternating $\mathscr{F}(M)$-k-multilinear maps α: $\mathscr{X}(M)\times\cdots\times\mathscr{X}(M)\to \mathscr{F}(M)'$, and as generalized quantities have the usual pointwise convergence topology. In fact, ψ is an isomorphism and a homeomorphism, and $\psi(\alpha)(X_1,\dots,X_k)=i_{X_k}\dots i_{X_1}\alpha/k!$ (see 3.4).

Proof. By use of a partition of unity, it is sufficient to prove the result locally. Clearly, if $\alpha\in\Omega^k(M)'$, then $\psi\alpha\in L_a^{k'}$. Also,

$$\varphi(\alpha(X_1,\dots,X_k))=\varphi\frac{1}{k!}(i_{X_k}\dots i_{X_1}\alpha)$$

$$=\frac{1}{k!}\,i_{X_k}\dots i_{X_1}\varphi(\alpha)$$

by 3.4, so that ψ as defined on $\Omega^k(M)'$ is an extension of ψ on $\Omega^k(M)$. As usual, φ denotes the natural embedding.

Also notice that for $\alpha\in\Omega^k(M)'$, $\omega\in\Omega_c^n(M)$, we have

$$(\psi\alpha)(X_1,\dots,X_k)\cdot\omega=\frac{1}{k!}\alpha(i_{X_k}\dots i_{X_1}\omega).$$

From this formula, ψ is clearly continuous. Also, ψ is one-to-one as $i_{X_k}\dots i_{X_1}\omega$ span all the $n-k$ forms.

To show ψ is onto, let $\rho\in L_a^{k'}$ and define α by

$$\alpha(i_{X_k}\dots i_{X_1}\omega)=k!\,\rho(X_1,\dots,X_k)\cdot\omega$$

where $X_k,\dots,X_1$ are basis vectors and extend by linearity. If $\rho_i\to\rho$, then $\alpha_i\to\alpha$ so that $\alpha\in\Omega^k(M)'$ and ψ^{-1} is continuous. $\square$

Therefore, in the language of $L_a^{k'}$ all the structure of $\Omega^k(M)'$ carries over, and we have, by continuity, the following basic formulae. (By abuse, we identify $\Omega^k(M)'$ and $L_a^{k'}$.)

4.2. Proposition. *Let $\alpha\in\Omega^k(M)'$, $\beta\in\Omega^l(M)$ and $X\in\mathscr{X}(M)$. Then we have*

(i) $\alpha\wedge\beta(X_1,\dots,X_{k+l})$

$$=\sum(\operatorname{sign}\pi)\,\alpha(X_{\pi(1)},\dots,X_{\pi(k)})\,\beta(X_{\pi(k+1)},\dots,X_{\pi(k+l)})/(k+l)!,$$

the sum being over all permutations π;

(ii) $(L_X\alpha)(X_1,\dots,X_k)$

$$=L_X[\alpha(X_1,\dots,X_k)]-\sum_{i=1}^{k}\alpha(X_1,\dots,[X,X_i],\dots,X_k),$$

(iii) $$d\alpha(K_0,\dots,X_k)=\Big[\sum_{i=0}^{k}(-1)^i L_{X_i}\alpha(X_0,\dots,\hat{X}_i,\dots,X_k)$$
$$+\sum_{0\le i<j\le k}(-1)^{i+j}\alpha([X_i,X_j],X_0,\dots,\hat{X}_j,\dots,X_k)\Big]\Big/(k+1)$$

where $\hat{X}_i$ denotes that X_i is omitted.

Proposition 4.1 motivates this:

4.3. Definition. *Let $\mathscr{T}_s^r(M)$ denote the tensorfields of type (r, s) on M. A* **generalized tensorfield** *of type (r, s) on M is an $\mathscr{F}(M)$-multilinear mapping*

$$t\colon \mathscr{X}^*(M)\times\cdots\times\mathscr{X}^*(M)\times\mathscr{X}(M)\times\cdots\times\mathscr{X}(M)\to\mathscr{F}(M)'$$

where r copies of $\mathscr{X}^*(M)$ and s of $\mathscr{X}(M)$ appear. Generalized tensorfields are denoted $\mathscr{T}_s^r(M)'$.

In particular, $\mathscr{X}(M)'=\mathscr{T}_0{}^1(M)'$ are **generalized vectorfields.** *($\mathscr{X}^*(M)'$ is identified with $\Omega^1(M)'$ by 4.1.)*

On $\mathscr{T}_s^r(M)'$ we put the topology of pointwise convergence, so that $t_i\to t$ iff

$$t_i(\alpha_1,\ldots,\alpha_r,X_1,\ldots,X_s)\cdot\omega\to t(\alpha_1,\ldots,\alpha_r,X_1,\ldots,X_s)\cdot\omega$$

for all $\alpha_i\in\mathscr{X}^*(M)$, $X_j\in\mathscr{X}(M)$ and $\omega\in\Omega_c^n(M)$.

An alternative way to define $\mathscr{T}_s^r(M)'$ is by means of generalized quantities (§ 1) as the next proposition shows.

4.4. Proposition. *Let $L_a^k(M)^*$ denote the $\mathscr{F}(M)$-multilinear maps*

$$t\colon \mathscr{X}^*(M)\times\cdots\times\mathscr{X}^*(M)\times\mathscr{X}(M)\times\cdots\times\mathscr{X}(M)\to\Omega_c^n(M)^*$$

with the pointwise convergence topology. Define $\varphi\colon \mathscr{T}_s^r(M)\to L_a^k(M)^*$ in the obvious way (2.1). Then φ is a monomorphism, and $\mathscr{T}_s^r(M)'$ defined above are exactly the generalized $\mathscr{T}_s^r(M)$ quantities. In particular, $\mathscr{T}_s^r(M)'$ is sequentially closed in $L_a^k(M)^*$.

The proof is a simple modification of 4.1 and so is omitted.

We can define support, singular support, smooth *etc.* as in § 2, and the same elementary properties hold.

Similarly, $\otimes$ has a unique extension

$$\otimes\colon \mathscr{T}_s{}^r(M)'\times\mathscr{T}_q{}^p(M)\to\mathscr{T}_{s+q}^{r+p}(M)',$$

and the action of an orientation preserving diffeomorphism $F\colon M\to N$ extends to a map:

$$F_*\colon \mathscr{T}_s{}^r(N)'\to\mathscr{T}_s{}^r(M)'.$$

The Lie derivative also extends to a map:

$$L_X\colon \mathscr{T}_s{}^r(M)'\to\mathscr{T}_s{}^r(M)'$$

for $X\in\mathscr{X}(M)$.

Also, in local coordinates a tensor has the usual expansion only with generalized coefficients, instead of smooth ones.

After this extension it is not hard to guess or to prove what properties hold. For example,

$$L_X(t\otimes t')=(L_X t)\otimes t'+t\otimes L_X t,$$

$$F^*(L_X t)=L_{F^*X}F^*t,$$

$$(F\circ G)^*=F^*\circ G^*$$

and if X has flow F_t,

$$F_{\tau *} L_X t = \frac{d}{dt}(F_{t*} t) \quad \text{at} \quad t=\tau.$$

We have made no *systematic* attempt at generalizing classical differential geometry, such as connections *etc.* Hints as to how this might be done are given in the appendix to § 10. See also MARSDEN [5].

§ 5. Generalized Vectorfields

The goals of this section are an extension of the Lie derivative to generalized vectorfields and their characterization as generalized derivations. We also prove a mild extension of GAUSS' divergence theorem.

5.1. Definition. *Let D denote the set of derivations*

$$\theta\colon \mathscr{F}(M) \to \mathscr{F}(M)$$

(*which is isomorphic to $\mathfrak{X}(M)$*; ABRAHAM [1, § 8]), *and D^* the derivations*

$$\theta\colon \mathscr{F}(M) \to \Omega_c^n(M)^*$$

with the pointwise convergence topology. Define $\varphi\colon D \to D^*$ in the obvious way (2.1). *Generalized D-quantities are called* **generalized derivations** *and are denoted D'* (see § 1).

Clearly D' consists of derivations $\theta\colon \mathscr{F}(M) \to \mathscr{F}(M)'$ but *need* not be all of them.

Then we have:

5.2. Theorem. *The map $L\colon \mathfrak{X}(M) \to D$; $X \mapsto L_X$ has a unique continuous extension $L\colon \mathfrak{X}(M)' \to D'$ and is an isomorphism and a homeomorphism. In fact, $L_X(f) = X(df) \in \mathscr{F}(M)'$.*

Proof. It follows at once from the definitions that L maps into D' and is continuous and $\mathscr{F}(M)$-linear.

Also, L is clearly one-to-one, for $X(df)=0$ for all $f \in \mathscr{F}(M)$ implies X is zero locally (using a basis) and hence globally.

To show L is onto, suppose $\theta_i \to \theta$ and $\theta_i \in D$. Let $\theta_i = L_{X_i}$ for $X_i \in \mathfrak{X}(M)$. We claim X_i converge in $\mathfrak{X}(M)'$. But this is clear locally using coordinates, and hence globally. If X is the limit, obviously $\theta = L_X$ (see similar situations in § 3). By a similar argument it follows that L^{-1} is continuous. □

This theorem allows us to define the Lie bracket (a direct definition in terms of coordinates is also possible). As usual we often write $X \cdot f$ for $L_X f$.

5.3. Theorem. *The Lie bracket has a unique extension to a map $[\,,\,]\colon \mathfrak{X}(M)' \times \mathfrak{X}(M) \to \mathfrak{X}(M)'$ continuous in the first variable. In fact,*

$$L_{[X,Y]} f = L_X(L_Y f) - L_Y(L_X f) \in \mathscr{F}(M)'$$

and the Jacobi identity

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

holds; $X \in \mathfrak{X}(M)'$, $Y, Z \in \mathfrak{X}(M)$.

Proof. Consider the derivation

$$\theta(f) = L_X L_Y f - L_Y L_X f.$$

We claim $\theta \in D'$. Suppose $X_i \to X$. Then since L_Y is continuous (3.3), $[L_{X_i}, L_Y] \to \theta$. Thus θ defines a vectorfield $[X, Y] \in \mathscr{X}(M)'$, continuous in X by 5.2. The theorem follows. □

Also by continuity, observe that if $F: M \to N$ is an (orientation preserving) diffeomorphism, then

$$F^*[X, Y] = [F^* X, F^* Y].$$

(Here $F^* X$ is given by $(F^* X)(f) = F^*(X(F_* f))$.)

In local coordinates, if $X = \sum X^i \, \partial/\partial x^i$; $X^i \in \mathscr{F}(U)'$ and $Y = \sum Y^i \, \partial/\partial x^i$; $Y^i \in \mathscr{F}(U)$, then

$$[X, Y] = \sum_{i,j} (X^i \, \partial Y^j/\partial x^i - Y^i \, \partial X^j/\partial x^i) \, \partial/\partial x^j.$$

5.4. Theorem. *Let $\alpha \in \Omega^k(M)$, $k \geqq 0$ and consider the map $\mathbf{i}: \mathscr{X}(M) \to \Omega^{k-1}(M)$; $X \mapsto \mathbf{i}_X \alpha$. Then $\mathbf{i}$ has a unique continuous extension*

$$\mathbf{i}: \mathscr{X}(M)' \to \Omega^{k-1}(M)'.$$

In fact, $\mathbf{i}_X \alpha(X_2, \dots, X_k) = X(\alpha_{X_2 \dots X_k}) \in \mathscr{F}(M)'$ where

$$\alpha_{X_2 \dots X_k}(Y) = \alpha(Y, X_2, \dots, X_k).$$

We have, in addition, for each $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $X \in \mathscr{X}(M)'$,

(i) $\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X \beta$;

(ii) $\mathbf{i}_X \alpha = X(\alpha)$ *if* $k = 1$, *and* $\mathbf{i}_X \alpha = 0$ *if* $k = 0$; *and*

(iii) $\mathbf{i}_{[X,Y]} = \mathbf{i}_X L_Y - L_Y \mathbf{i}_X$ *for* $Y \in \mathscr{X}(M)$.

Proof. Merely observe that $\mathbf{i}_X$ so defined is continuous in X and coincides with the usual inner product if X is smooth. The rest holds by continuity. □

In a similar way we may prove:

5.5. Theorem. *Let $\alpha \in \Omega^k(M)$ and $\mathbf{L}: \mathscr{X}(M) \to \Omega^k(M)$; $X \mapsto \mathbf{L}_X \alpha$. Then $\mathbf{L}$ has a unique continuous extension $\mathbf{L}: \mathscr{X}(M)' \to \Omega^k(M)'$. In fact, $\mathbf{L}_X \alpha = \mathbf{d}\, \mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d} \alpha$, and*

$$(\mathbf{L}_X \alpha)(X_1, \dots, X_k) = \mathbf{L}_X(\alpha(X_1, \dots, X_k)) - \sum_{i=1}^{k} \alpha(X_1, \dots, [X, X_i], \dots, X_k).$$

Moreover,

(i) $\mathbf{L}_X f = X(\mathbf{d} f)$;

(ii) $\mathbf{L}_X \mathbf{d} = \mathbf{d} \mathbf{L}_X$;

(iii) $\mathbf{L}_X(\alpha \wedge \beta) = (\mathbf{L}_X \alpha) \wedge \beta + \alpha \wedge \mathbf{L}_X \beta$;

(iv) $\mathbf{L}_{[X,Y]} \alpha = \mathbf{L}_X \mathbf{L}_Y \alpha - \mathbf{L}_Y \mathbf{L}_X \alpha$;

(v) *if $F: M \to N$ is an (orientation preserving) diffeomorphism* $F^*(\mathbf{L}_X \alpha) = \mathbf{L}_{F^* X} F^* \alpha$;

(vi) $\mathbf{L}_{fX} \alpha = f \mathbf{L}_X \alpha + (\mathbf{d} f) \wedge \mathbf{i}_X \alpha$;

(vii) $\mathbf{i}_{[X\ Y]} = \mathbf{L}_X \mathbf{i}_Y - \mathbf{i}_Y \mathbf{L}_X$, *where* $\alpha\in\Omega^k(M)$, $X\in\mathscr{X}(M)'$, $Y\in\mathscr{X}(M)$, $f\in\mathscr{F}(M)$. (*It is also easy to see that* (i)–(iii) *determine* $L_X\colon\Omega^k(M)\to\Omega^k(M)'$ *uniquely.*)

We also have a unique extension

$$L_X\colon\ \mathscr{T}^r_s(M)\to\mathscr{T}^r_s(M)'$$

and is a tensor derivation. (See § 4.)

Finally in this section we consider divergence:

5.6. Definition. *Let* $X\in\mathscr{X}(M)'$ *and* Ω *be a volume on* M. *Then* $\operatorname{div}_\Omega X\in\mathscr{F}(M)'$ *is defined by*

$$L_X\,\Omega=(\operatorname{div}_\Omega X)\,\Omega$$

and is called the **divergence** *of* X *with respect to* Ω. *(Of course,* $\operatorname{div}_\Omega X$ *is uniquely determined.)*

In a local chart, if $\Omega=dx^1\wedge\cdots\wedge dx^n$, then

$$\operatorname{div}_\Omega X=\sum_{i=1}^{n}\frac{\partial X^i}{\partial x^i}.$$

Also, by continuity, we have for $X\in\mathscr{X}(M)'$ the following:

(i) if $f(m)\neq 0$ for all $m\in M$, $f\in\mathscr{F}(M)$,

$$\operatorname{div}_{f\Omega}X=\operatorname{div}_\Omega X+(L_X f)/f;$$

(ii) $\operatorname{div}_\Omega(gX)=g\operatorname{div}_\Omega X+L_X g$; $g\in\mathscr{F}(M)$.

For the proofs in the smooth case, which also hold here, see ABRAHAM [1, p. 77].

STOKES' theorem (ABRAHAM [1, p. 82]) has the following mild generalization.

5.7. Proposition. *Let* $(V, V_0, \operatorname{bd} V_0)$ *be a compact orientable manifold with boundary and* $\alpha\in\Omega^{n-1}(V)'$ *have singular support* $C\subset V_0$. *Then* $d\alpha$ *is integrable on* V_0, *and*

$$\int_{V_0} d\alpha=\int_{\operatorname{bd} V_0} i_*\alpha$$

where $i\colon \operatorname{bd} V_0\to V$ *is the inclusion map.*

Proof. First, let $\omega\in\Omega^{n-1}(V)'$ and

$$C=\operatorname{sing\,supp}\omega\subset V_0,$$

then we claim that ω is integrable on V_0. Let $\{g_i\}$ be any partition of unity and h the sum of g_i with supports intersecting C, the sum being finite as C is compact. Hence

$$\sum\omega(g_i)=\omega(h)+\int(1-h)\,\omega$$

converges, as $(1-h)\,\omega$ is smooth. If $\{g_i'\}$ is another partition of unity,

$$\int(1-h)\,\omega-\int(1-h')\,\omega=\int(h'-h)\,\omega$$

and also

$$\int h\,\omega-\int h'\,\omega=\int(h-h')\,\omega$$

so that ω is integrable. In particular, $d\alpha$ is integrable. For the theorem, we have, by definition:

$$\int h\, d\alpha = -\int (dh)\wedge\alpha$$

since $\int d(h\alpha)=0$ by 3.2 (iii). On the other hand,

$$(1-h)\,d\alpha = d(1-h)\alpha) + (dh)\wedge\alpha$$

both terms of which are smooth as $d\alpha=0$ on C. Therefore

$$\int_{V_0} d\alpha = \int_{V_0} d((1-h)\alpha)$$
$$= \int_{\mathrm{bd}\,V_0} i_*(1-h)\,\alpha = \int_{\mathrm{bd}\,V_0} i_*\alpha$$

as $h=0$ on $\mathrm{bd}\,V_0$, using STOKES' theorem. $\square$

From this we have **Gauss' divergence theorem:**

5.8. Corollary. *Let $(V, V_0, \mathrm{bd}\,V_0)$ be a compact orientable manifold with boundary and $X\in\mathscr{X}(M)'$ have singular support $C\subset V_0$. If Ω is a volume on V, then*

$$\int_{V_0} (\mathrm{div}_\Omega X)\,\Omega = \int_{\mathrm{bd}\,V_0} i_*(i_X\Omega).$$

In particular if X is incompressible $(\mathrm{div}_\Omega X=0)$, *then*

$$\int_{\mathrm{bd}\,V_0} i_*(i_X\Omega)=0.$$

Proof. This is an immediate consequence of 5.7 and the fact that $(\mathrm{div}_\Omega X)\Omega = L_X\Omega = d\, i_X\Omega$. $\square$

§ 6. Flows

As we saw in the Introduction, there is a clear physical need for assigning a flow to a generalized vectorfield. This is the central problem of this section.

To motivate the approach, we consider an example. On R^2 consider the generalized vectorfield

$$X(q,p) = p\frac{\partial}{\partial q} - \delta(q)\frac{\partial}{\partial p}$$

for $(p,q)\in R\times R$, and δ the delta function. As we shall see later, this is the vectorfield associated with the Hamiltonian,

$$H(q,p)=\tfrac{1}{2}p^2 + V(q)$$

where $V(q)=1$ if $q\geqq 0$ and $V(q)=0$ if $q<0$. Now approximating H by smooth functions $H_t\to H$, we see that the flows converge almost everywhere to the discontinuous flow one expects from high school physics. The flow is energy and measure preserving.

Classically, following KOOPMAN, we would properly view this flow as a unitary flow on $L^2(R^2)$. If the flow were smooth, we would expect the infinitesimal generator to be just X itself, but this is not true here. In otherwords, *one cannot shortcut* *the problem of flows using Stone's theorem.* We shall say a little more on this point below. (Another possible method would be the use of the Trotter-Kato theorem (YOSIDA [1, p. 272]), although this is more complicated than our method and leads to severe technicalities.)

6.1. Definition. *Let X be a generalized vectorfield on M and X_i smooth vectorfields with complete flows F_t^i, and $X_i \to X$ in $\mathfrak{X}(M)'$. We say that X* **has a (measureable) flow** *F_t iff*

(i) *$F_t^i \to F_t$ almost everywhere for each $t \in \mathbf{R}$;*

(ii) *for each $t \in \mathbf{R}$ and each compact set $C \subset M$, there is a compact set $K \subset M$ with $C \subset K$ and $F_t^i(C) \subset K$ for all $i = 1, 2, \ldots$.*

One may similarly define a local flow. Standard examples of non-Lipschitz vectorfields show that the flow need not be unique; therefore to specify a flow one needs to specify a sequence X_i satisfying 6.1. For the Hamiltonian case there is usually a natural way to do this. The question of existence is solved in 6.3.

6.2. Theorem. *If F_t satisfies* (i) *and* (ii) *of* 6.1 *then F_t is automatically a flow; that is, $F_{t+s} = F_t \circ F_s$ almost everywhere for each $t, s \in \mathbf{R}$.*

Proof. Clearly $F_s^i \circ F_t^i \to F_{s+t}$ a.e. Let $C \subset M$ be compact and choose K_1 and K_2 compact so that $F_t^i(C) \subset K_1$ and $F_s^i(K_1) \subset K_2$ for all i. By Egoroff's theorem, for any $\varepsilon > 0$, there is a set $A \subset K_1$ with $\mu(A) < \varepsilon$ (by use of some smooth measure), such that $F_s^i \to F_s$ uniformly on $C \backslash A$ and $F_t^i \to F_t$ uniformly on $K_1 \backslash A$, the uniformity being with respect to some metric on M. It follows easily that $F_s^i \circ F_t^i \to F_s \circ F_t$ pointwise on $C \backslash A$, and so $F_{s+t} = F_s \circ F_t$ on $C \backslash A$. Since this holds on the union of these sets, it holds almost everywhere. □

The basic existence theorem is an follows:

6.3. Theorem. *Let Ω be a volume on M and $X \in \mathfrak{X}(M)'$. Suppose that*

(i) *X has compact support;*

(ii) sing supp X *has measure zero;*

(iii) *there are $X_i \to X$ all with supports in some compact set and for a sequence of open sets $U_i \downarrow$* sing supp X, *$X_i = X$ outside U_i;*

(iv) $\mathrm{div}_\Omega X_i$ *are uniformly bounded.*

Then X has a flow. In fact, if F_t^i is the flow of X_i, some subsequence of F_t^i converges as in 6.1.

Since the theorem is designed for the Hamiltonian case, the proof will be postponed until 8.4.

The hypothesis (i) is no restriction; it is used so the flow will be complete. Otherwise, we would obtain only a **local flow**; more precisely, if $h = 1$ on an open set U and has compact support, the flow of hX on U is the local flow of X.

The flow F_t of X will be smooth outside sing supp X and in general suffers a discontinuity as it "passes over" the singular support. Notice that an equation like

$$X(m) = \frac{d}{dt} F_t(m), \qquad t = 0$$

or

$$X(f)=\frac{d}{dt}F_{t*}f, \qquad t=0$$

will hold only off sing supp X. But we do have

$$X(f)=\operatorname*{limit}_{l\to\infty}\frac{d}{dt}(F_{t*}^{l}f) \qquad (t=0);$$

see, however, 6.5.

The next result will be quite useful in the Hamiltonian case.

6.4. Proposition. (i) *In* 6.1, *suppose* $X_l(f_l)=0$ *and* $f_l\to f$ *a.e., for* $f_l\in\mathscr{F}(m)$; *then* $f\circ F_t=f$ *a.e. for all* $t\in R$;

(ii) *if, in* 6.1, $\operatorname{div}_\Omega X_l=0$, *then* $\operatorname{div}_\Omega X=0$ *and* F_t *is measure preserving;*

(iii) *if* $X_l(f_l)=0$ *and* $f_l\to f$ *a.e. off* sing supp f *for* $f\in\mathscr{F}(M)'$, *then* $f\circ F_t=f$ *a.e. on the set (where it makes sense)*

$$C_t=\{m\in M:\ m\notin \operatorname{sing\,supp} f,\ F_t(m)\notin \operatorname{sing\,supp} f\}.$$

Proof. (i) We have $f_l\circ F_t{}^l=f_l$. Now argue as in 6.2 by Egoroff's theorem. The proof of (iii) is similar.

For (ii), let $A\subset M$ lie in some compact set and have characteristic function C_A. Then, using (ii) of 6.1, we have by bounded convergence (take A a disc, say)

$$\mu_\Omega(A)=\operatorname*{limit}_{l\to\infty}\int C_A\circ F_t{}^l d\mu_\Omega=\int C_A\circ F_t\, d\mu_\Omega.$$

This proves the assertion. □

6.5. Proposition. *In* 6.1, *suppose X is locally bounded and* $X_l\to X$ *a.e. and locally boundedly by an integrable function (that is, for each* $f\in\mathscr{F}(M)$, $X_l(f)\to X(f)$ *a.e. and locally dominated).*

Then for each $f\in\mathscr{F}(M)$, *the generalized derivative of the map* $t\mapsto f\circ F_t$ *is equal to* $L_X f\circ F_t$. *Briefly, for all* $t\in R$, *we have*

$$L_X f\circ F_t=\frac{d}{dt}(F_{t*}f).$$

In particular, we have $L_X f=0$ *implies* $f\circ F_t=f$ *a.e.*

Proof. Let $\omega\in\Omega_c^n(M)$ and $t\in R$. Then

$$\operatorname*{limit}_{l\to\infty}\int F_{t*}^{l}(L_{X_l}f)\cdot\omega=\int F_{t*}(L_X f)\cdot\omega$$

by the dominated convergence theorem. But also this equals

$$\operatorname*{limit}_{l\to\infty}\frac{d}{dt}\int F_{t*}^{l}f\cdot\omega=\frac{d}{dt}\operatorname*{limit}_{l\to\infty}\int F_{t*}^{l}f\cdot\omega$$

since the generalized derivative is continuous. This proves the initial claim, and 6.5 follows. □

In this proof we have made use of the fact that if f, g are locally integrable and equal in $\mathscr{F}(M)'$, then they are equal a.e., which was proven following 2.1.

Notice that 6.5 does not make sense if the hypothesis of local integrability is removed. Also, the techniques may be used to give an alternative proof of 6.4.

Finally we make a few remarks on the connection with Koopmanism. First, fix a volume Ω on M and suppose $\operatorname{div}_\Omega X = 0$. If X is smooth with complete flow F_t, then there is induced a unitary flow U_t on $L^2(M)$. The infinitesimal generator of this flow is an extension of X, acting on $\mathscr{F}_c(M)$. On the other hand if iX has a self-adjoint extension (say $X(\mathscr{F}_c(M)) \subset L^2(m)$), there will be a corresponding flow U_t and F_t. (See Halmos [3, p. 42–45].) However, this procedure is extremely limited, and this is the reason for adopting 6.1. (We do not know if F_t constructed this way is a flow in the sense of 6.1, or conversely.) In any event, if we are given X (or the Hamiltonian) as a distribution we cannot compute the infinitesimal generator in L^2 until we know the flow!

Appendix: Closed Orbits

A basic theorem of the smooth theory is that an orbit is periodic iff it is compact. Of course this is false for measurable flows.

6.6. Proposition. *In* 6.1 *suppose for some* $m \in M$ *that*

(i) $F_t^i(m) \to F_t(m)$ (*for all* $t \in R$), *and* $F_t(m)$ *is continuous for almost all* $t \in R$;

(ii) $F_t^i(m)$, *or a subsequence are closed orbits with bounded periods*, $i = 1, 2, 3, \ldots$.

Then there is a $\tau \geqq 0$ *such that for almost all* $t \in R$, $F_{t+\tau}(m) = F_t(m)$.

Proof. Let F_t^i have period τ_i and taking subsequences, we may assume $\tau_i \to \tau$. By Egoroff's theorem, there is a set $A \subset R$ of measure $< \varepsilon$ such that $F_t^i(m) \to F_t(m)$ uniformly off A, but in some interval containing all the periods. For $t \notin A$,

$$F_{t+\tau}(m) = \lim_{i \to \infty} F_{t+\tau}^i(m) = \lim_{i \to \infty} F_{t+\tau-\tau_i}^i(m).$$

But if t is a point of continuity of $F_t(m)$, this equals $F_t(m)$ in view of the following inequality, where d is a metric on M;

$$d\big(F_{t+\tau-\tau_i}^i(m), F_t(m)\big) \leqq d\big(F_{t+\tau-\tau_i}(m), F_{t+\tau-\tau_i}(m) + d\big(F_{t+\tau-\tau_i}(m), F_t(m)\big).$$

Therefore we conclude that $F_{t+\tau}(m) = F_t(m)$ for all t except on a set of measure $< \varepsilon$. Thus it holds almost everywhere. □

There is a variety of closed orbit theorems for smooth Hamiltonian flows to aid in fulfilling the conditions of the theorem. The most important of these are probably Abraham's closed orbit theorem (Abraham [1, p. 178]), the Liapounov-Kelley theorem (Abraham [1, p. 180]) and Arnold's theorem (Abraham [1, p. 112] and Arnold-Avez [1, p. 182]). We add a remark which is restricted to the two-dimensional case. (It should not be confused with Arnold's theorem as we allow critical points.)

The result is a follows (for this proposition only we assume a knowledge of smooth Hamiltonian flows and the Morse lemma):

6.7. Proposition. *Let M be a two-dimensional symplectic manifold and $H \in \mathscr{F}(M)$. Suppose H has non degenerate critical points and its orbits are bounded. Then every orbit is a closed orbit (or critical point) iff the critical points are of index zero or two* (i.e., *are maxima or minima*).

Proof. By the Morse lemma, if m_0 is a critical point of index 0, then there is a local chart in which

$$H(x, y) = H(m_0) + x^2 + y^2.$$

Thus, in either case the points in a neighborhood of every critical point lie in a regular energy surface, and hence all points lie in a regular energy surface. But an orbit in a regular energy surface which is bounded is closed (use ABRAHAM [1, p. 40]). The converse is equally clear. □

Chapter Two: Hamiltonian Systems

§ 7. Symplectic Geometry

The basic setting of Hamiltonian mechanics, smooth or generalized, is a symplectic manifold. This section extends the basic operations to the generalized case.

Recall that a symplectic manifold (M, ω) consists of a (finite dimensional) manifold M and a non-degenerate closed two form ω. Here ω is smooth.

The basic structure theorem is as follows:

7.1. Theorem (DARBOUX). *Let (M, ω) be a symplectic manifold. Then M is even dimensional, say $2n$, and for each $m \in M$ there is a coordinate chart $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that, locally,*

$$\omega = \sum_{i=1}^{n} d q^i \wedge d p_i .$$

For the proof, see ABRAHAM [1, § 14].

The most important example of a symplectic manifold is the cotangent bundle of a manifold with the natural symplectic structure; ABRAHAM [1, p. 96]. We shall deal with this case explicitly in § 9.

It is meaningful to talk about generalized symplectic forms although this does not lead to a satisfactory theory. Clearly Darboux's theorem *cannot* hold in that case.

Recall that if (M, ω) is a symplectic manifold, we define $\omega_\flat : \mathscr{X}(M) \to \mathscr{X}^*(M)$; $\omega_\flat(X) \cdot Y = \omega(X, Y)$, and it is an isomorphism. Its inverse is denoted $\omega_\sharp$, and we put $X^\flat = 2\omega_\flat(X)$, $\alpha^\sharp = \frac{1}{2}\omega_\sharp(\alpha)$.

7.2. Proposition. *The maps $X \mapsto X^\flat$ and $\alpha \mapsto \alpha^\sharp$ have unique continuous extensions to maps;*

$$^\flat : \quad \mathscr{X}(M)' \to \mathscr{X}^*(M)'$$

$$^\sharp : \ \mathscr{X}^*(M)' \to \mathscr{X}(M)'.$$

These are homeomorphisms and isomorphisms and are inverse of each other. In fact, $X^\flat(Y) = -X(Y^\flat) \in \mathscr{F}(M)'$, and $\alpha^\sharp(\beta) = -\alpha(\beta^\sharp) \in \mathscr{F}(M)'$ for $Y \in \mathscr{X}(M)$, $\beta \in \mathscr{X}^*(M)$. Also, $X^\flat = i_X \omega$ (see 3.4).

Proof. Consider the maps so defined. First, they extend the usual ones, for if $X \in \mathscr{X}(M)$, then

$$\varphi(X)^\flat \cdot Y = -\varphi(X) \cdot Y^\flat = -\varphi(X(Y^\flat))$$

where $\varphi: \mathscr{X}(M) \to \mathscr{X}(M)'$ is the embedding. But

$$X(Y^\flat) = 2\omega \cdot (Y, X) = -X^\flat(Y),$$

so that

$$\varphi(X)^\flat \cdot Y = \varphi(X^\flat) \cdot Y$$

proving the contention. Clearly $^\flat$, $^\sharp$ map into $\mathscr{X}^*(M)'$ and $\mathscr{X}(M)'$ and are continuous and inverses of one another. The statement $X^\flat = i_X \omega$ follows by continuity, or directly. $\square$

Of particular interest is the map

$$H \mapsto X_H = (dH)^\sharp$$

which extends as well:

7.3. Proposition. *The map $X: \mathscr{F}(M) \to \mathscr{X}(M)$; $H \mapsto X_H$ has a unique continuous extension $X: \mathscr{F}(M)' \to \mathscr{X}(M)'$. It is $H \mapsto X_H = (dH)^\sharp$.*

Proof. Clear from 7.3 and 3.2. $\square$

In particular, if $H_i \to H$ then $X_{H_i} \to X_H$. If $H \in \mathscr{F}(M)'$, then in a canonical chart (given by 7.1)

$$X_H = \sum_{i=1}^{n} \left\{ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right\}.$$

Recall that the Poisson bracket is given by

$$\{\alpha, \beta\} = -[\alpha^\sharp, \beta^\sharp]^\flat \qquad \text{for } \alpha, \beta \in \mathscr{X}^*(M)$$

and

$$\{f, g\} = -i_{X_f} i_{X_g} \omega = L_{X_g} f = -L_{X_f} g; \quad f, g \in \mathscr{F}(M).$$

From ABRAHAM [1, p. 98] we see that $\{\alpha, dH\} = L_{X_H} \alpha$.

7.4. Proposition. *The Poisson brackets have unique extensions to maps continuous in the first variable:*

$$\{\ ,\ \}: \mathscr{X}^*(M)' \times \mathscr{X}^*(M) \to \mathscr{X}^*(M)'$$

$$\{\ ,\ \}: \mathscr{F}(M)' \times \mathscr{F}(M) \to \mathscr{F}(M)'.$$

The proof is clear. By continuity or an easy computation, essentially all the basic formulae carry over. We list a few for reference:

7.5. Proposition. *Let $f \in \mathscr{F}(M)'$, $g \in \mathscr{F}(M)$, $\alpha \in \mathscr{X}^*(M)'$, and $\beta \in \mathscr{X}^*(M)$. Then*

(i) $\{\alpha, \beta\} = -L_{\alpha^\sharp} \beta + L_{\beta^\sharp} \alpha + d(i_{\alpha^\sharp} i_{\beta^\sharp} \omega)$ *and* $\{df, \beta\} = -L_{X_f} \beta$;

(ii) $\{f, g\} = L_{X_g} f = -L_{X_f} g$;

(iii) $\{f, g\} \cdot \rho = -f(L_{X_g} \rho)$ for $\rho \in \Omega_c^n(M)$;

(iv) *if Ω_ω is the standard volume given by*

$$(-1)^r (\omega \wedge \cdots \wedge \omega)/n!, \qquad r = n(n-1)/2,$$

then

$$\{f, g\} \cdot (h\,\Omega_\omega) = f(\{g, h\}\,\Omega_\omega); \qquad h \in \mathscr{F}_c(M);$$

(v) *in a canonical chart,*

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right);$$

(vi) $\{df, dg\} = d\{f, g\}$;

(vii) $L_{X_f}\,\omega = 0$;

(viii) $X_{\{f,g\}} = -[X_f, X_g]$.

Notice that by continuity we also have that a diffeomorphism $F: M \to M$ is symplectic, that is, $F_*\omega = \omega$, iff $F_* X_H = X_{F_* H}$ for all $H \in \mathscr{F}(M)'$ iff $F_*\{f, g\} = \{F_* f, F_* g\}$ for all $f \in \mathscr{F}(M)'$, $g \in \mathscr{F}(M)$.

Some other theorems do not carry over in full generality. For example, the Poincaré-Cartan theorem (ABRAHAM [1, p. 103]) holds only for functions and n-forms, as $F_*\alpha$ does not, in general, make sense. However, a little more of the relative integral invariant theorem can be recovered using 5.7.

§ 8. Hamiltonian Systems

The basic philosophy of Hamiltonian mechanics is that the Hamiltonian function should determine the time evolution of the system. This, in fact, motivated our treatment of § 6. All the machinery of preceding sections makes our job here particularly easy.

8.1. Theorem. *Let (M, ω) be a symplectic manifold and $X \in \mathscr{X}(M)'$. Then the following are equivalent:*

(i) $i_X\,\omega$ *is closed;*

(ii) $L_X\,\omega = 0$;

(iii) *for each $m \in M$ there is a neighborhood U of m so $X|U = X_H|U$ for some $H \in \mathscr{F}(M)'$;*

(iv) *locally, there is an $H \in \mathscr{F}(U)'$ such that for each $f \in \mathscr{F}(U)$, $L_X f = \{f, H\}$;*

(v) *locally, there is an $H \in \mathscr{F}(U)'$ such that for each $\alpha \in \mathscr{X}^*(M)$, $L_X \alpha = \{\alpha, dH\}$.*

The proof is clear by the generalized Poincaré lemma 3.6. As usual, a vectorfield satisfying 8.1 is called **locally Hamiltonian,** and is (globally) **Hamiltonian** if $X = X_H$ for some $H \in \mathscr{F}(M)'$.

8.2. Definition. *Let $X = X_H$ be a Hamiltonian vectorfield for $H \in \mathscr{F}(M)'$. We say X is* **Hamiltonian regular** *iff there exist $H_t \in \mathscr{F}(M)$ so $H_t \to H$ in $\mathscr{F}(M)'$, and $H_t \to H$ almost everywhere off* sing supp H *(and so $X_{H_t} \to X_H$ by* 7.3) *and X_{H_t} satisfy the flow conditions of* 6.1.

If sing supp H has measure zero (and dH has compact support), this always holds. See 8.4.

The basic conservation theorem is as follows (it applies to the example in § 6 for instance):

8.3. Theorem. *Suppose X_H, $H\in\mathscr{F}(M)'$ is Hamiltonian regular with flow F_t. Then*

(i) *F_t is measure preserving with respect to the standard volume (Liouville's theorem);*

(ii) *$H\circ F_t=H$ a.e. on the set (whose complement has measure zero)*

$$C_t=\{m\in M:\ m\notin \operatorname{sing\,supp} H;\ F_t(m)\notin \operatorname{sing\,supp} H\}.$$

This is an immediate consequence of 6.4 and the facts that

$$X_{H_i}(H_i)=\{H_i,H_i\}=0,\quad\text{and}\quad L_{X_{H_i}}\Omega_\omega=0\qquad(7.5\,(\text{vii})).$$

It is good to keep some examples in mind regarding 8.3. For instance, on $T^*R\approx R^2$ consider the Hamiltonian

$$H=\tfrac{1}{2}p^2+\delta(q).$$

This corresponds to reflection off a wall at the origin. Obviously it is Hamiltonian regular. Energy is conserved as long as we omit the origin. This is exactly what C_t in 8.3 does.

Notice that for $f,g\in\mathscr{F}(M)'$, $\{f,g\}$ is, in general meaningless, so that we *cannot* deduce conservation of energy from $\{H,H\}=0$, but require some limiting procedure as provided by 6.3.

From 5.8 note that for any $H\in\mathscr{F}(M)'$ with sing supp $H\subset V_0$, where $(M, V_0, \operatorname{bd}V_0)$ is a compact orientable manifold with boundary, we have

$$\int i_*(i_{X_H}\Omega_\omega)=0$$

since $\operatorname{div}_{\Omega_\omega}X_H=0$.

The usual elementary statements about constants of motion hold. First, suppose H is smooth, $f\in\mathscr{F}(M)'$ and $L_{X_H}f=0$, or $\{f,H\}=0$. Then $F_{t*}f=f$, or f is a constant of the motion, and conversely. This follows by the flow theorem 3.7. If $f\in\mathscr{F}(M)'$ and $g\in\mathscr{F}(M)$ are constants of the motion, so is $\{f,g\}$ by the Jacobi identity. Similar statements hold for one forms.

Dually, suppose $H\in\mathscr{F}(M)'$ and $L_{X_H}f=0$ for $f\in\mathscr{F}(M)$ and $L_{X_H}g=0$ for $g\in\mathscr{F}(M)$. Then $L_{X_H}\{f,g\}=0$. These may be interpreted as constants of the motion if 6.4 applies.

The systematic way of discovering constants of motion is given in the next section.

We now prove the basic existence theorem for flows, promised in § 6. For Hamiltonian vectorfields the theorem is as follows:

8.4. Theorem. *Let (M,ω) be a symplectic manifold and X_H a generalized Hamiltonian vectorfield on M with compact support and with singular support of measure zero. Then X_H possesses a Hamiltonian regular flow which is measure preserving* (8.2).

Proof. As the proof is somewhat involved, we first sketch the idea. The first step is to reinterpret flows as continuous maps from R to the measurable functions $f\colon M\to M$ with the metric of convergence in measure. Then in the framework of the Ascoli theorem we extract a convergent subsequence from the approximating flows. Finally a further subsequence is extracted which converges almost everywhere. In view of 6.2, this will suffice.

Let $C = \operatorname{sing\,supp} H = \operatorname{sing\,supp} X_H$, and let U_n be a sequence of open sets decreasing to C. Find $H_n \to H$, $H_n \in \mathscr{F}(M)$ with $H_n = H$ outside U_n and $\operatorname{supp}(dH_n) \subset K$ for a fixed compact set K.

Let $F_t^{(n)}$ denote the complete flow of X_{H_n}. Clearly $F_t^{(n)}$ satisfy (i), (ii) of 6.1. It thus is sufficient to show that some subsequence of $F_t^{(n)}$ converges almost everywhere. Also, it is enough to work in the compact set K.

In order to do this, consider the complete metric space A consisting of measurable maps $f: K \to K$ with the metric

$$d(f, g) = \inf\{r \in \mathbf{R}, r \geqq 0 \colon \mu\{x \in K \colon d(f(x), g(x)) > r\} \leqq r\}$$

where μ is the volume measure on M and d is some metric on M.

Let C denote the topological space of continuous maps $\alpha: \mathbf{R} \to A$ with the topology of uniform convergence on compact sets (see KELLEY [1, Ch. 7]). Let $F \subset C$ denote the subset of maps

$$t \mapsto F_t^{(n)}; \quad n = 0, 1, 2, \ldots.$$

We claim that there is an infinite subset of F which is relatively compact in C; that is, has compact closure. To see this, we verify the hypotheses of the classical Ascoli theorem (KELLEY [1, p. 233 and 239 ex. G]).

First we claim that F is uniformly equi-continuous. Equi-continuity at $t_0 = 0$ means for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|t| < \delta$ implies $d(F_t^{(n)}, I) < \varepsilon$ for all n, where I is the identity map. However this is clear since all $F_t^{(n)}$ equal a smooth flow on a compact set outside a set (U_n) of arbitrarily small measure. Now uniform equi-continuity is immediate since

$$d(F_t^{(n)}, F_{t_0}^{(n)}) = d(F_{t-t_0}^{(n)}, I),$$

by use of the semi-group property and the fact that each $F_t^{(n)}$ is measure preserving. (In the case of 6.3, here is where the uniform boundedness on the divergences is applied.)

Secondly, we claim that for each $t \in \mathbf{R}$ the maps $F_t^{(n)}$ have a convergent subsequence (in A). If this is not the case, we will obtain a contradiction by showing $\{F_t^{(n)}\}$ is totally bounded and hence compact. In fact, let $\delta > 0$ and V_n be a disc of radius δ about $F_t^{(n)}$ in A. It suffices to show a finite number of these cover $\{F_t^{(n)}\}$. If not, there is a subsequence whose members are a distance $\delta/2$ apart, at least. That is,

$$\mu\{x \in K \colon d(F_t^{(n)}(x), F_t^{(m)}(x)) > \delta/2\} > \delta/2.$$

Since K has finite measure, this means there is a further subsequence $F_t^{(m)}(x)$ for some $x \in K$ which does not have a convergent subsequence. This contradicts compactness of K.

Third, we claim that the subsequences chosen in the previous step may be assumed the same for all t. In fact, choose a subsequence common to all rationals t_k. Then if $t_k \to t$, we have

$$d(F_t^{(n)}, F_t^{(m)}) \leqq d(F_{t_k}^{(n)}, F_{t_k}^{(m)}) + d(F_{t_k}^{(n)}, F_t^{(n)}) + d(F_{t_k}^{(m)}, F_t^{(m)}).$$

Choose k large so that the last two terms are $<\varepsilon/3$ uniformly in n, and with this k fixed the first term is small for $n, m \geqq N$. Thus, this subsequence converges for all $t \in R$.

Thus, using the above subsequence for F, we see that F is relatively compact in C and is uniformly equi-continuous. Hence by the Ascoli theorem there is a subsequence which converges to an element F_t of C uniformly on compact sets.

For each t then, $F_t^{(n)}$ converges to F_t in measure. Since it is uniform for $|t| \leqq a$, we can find a single subsequence converging almost everywhere for $|t| \leqq a$ (using the proof in HALMOS [1, p. 93].) Choosing a an integer, we can find a single subsequence $F_t^{(n)}$ converging to F_t a.e. for all $t \in R$. This completes the proof. □

Note that we have, as a corollary, that F_t is continuous in t, using the metric of convergence in measure.

This theorem is basic to our presentation and justifies the intuitive feeling that if $H_i \to H$, the flows of X_{H_i} should converge (weakly); actually the theorem shows a subsequence may be necessary. More precisely, we have

8.5. Corollary. *Suppose, in* 8.4 *that for a sequence of open sets* $U_i \downarrow$ sing supp H *we have* $H_i = H$ *outside* U_i *and* $H_i \to H$ *in* $\mathscr{F}(M)'$. *Then for some subsequence of* H_i, X_H *is Hamiltonian regular.*

The most important setting for Hamiltonian mechanics, smooth or not, is on the cotangent bundle of a manifold, T^*M, which has an intrinsic symplectic structure. See ABRAHAM [1, § 14]). Locally, in the natural coordinates, it is given by the formula in 7.1.

In this setting, Hamiltonians are typically given by (locally)

$$H(q, p) = \tfrac{1}{2} \sum_{i,j} g^{ij}(q)\, p_i\, p_j + V(q)$$

where g arises from a Riemannian metric. It often occurs that g and V are not smooth, as we have seen. However, intuition tells us that *the flow should produce continuous curves in q-space, regardless of smoothness.* This is in fact true in our setting provided g are locally bounded functions and the approximating g converge dominated by a locally integrable function. In fact, by the dominated convergence theorem and the smooth equations of motion, we deduce

$$q^i(t) = \sum_j \int_0^t g^{ij}(q(s))\, p_j(s)\, ds$$

which proves the assertion. For further results along these lines see Section 10 and MARSDEN [5].

Appendix A: Symplectic Maps

We have seen that the flow of a generalized Hamiltonian vectorfield is volume preserving. It is reasonable to ask in what sense is it symplectic, or a canonical transformation?

8.6. Definition. *Let* (M, ω) *be a symplectic manifold and* $F\colon M \to M$ *measurable. We say* F *is* **symplectic** *iff there exist smooth symplectic maps* $F_i\colon M \to M$ *such that*

(i) $F_i(x) \to F(x)$ a.e.

(ii) *for each compact set $C \subset M$ there is a compact set $K \subset M$ so $F_i(C) \subset K$ for all i.*

Notice that a statement like $F_* \omega = \omega$ does not make sense in general, but here

$$\operatorname*{limit}_{i \to \infty} F_{i*} \omega = \omega .$$

8.7. Proposition. *Suppose F is symplectic. Then*

(i) $\operatorname*{limit}_{i \to \infty} \{F_{i*} f, F_{i*} g\} = \{f, g\} \circ F$;

(ii) $\operatorname*{limit}_{i \to \infty} F_{i*} X_f = X_{f \cdot F}$

for $f, g \in \mathscr{F}(M)$, the limit being in $\mathscr{F}(M)'$.

Proof. Note that $f \circ F_i \to f \circ F$ in $\mathscr{F}(M)'$. Since $F_i \to F$ a.e. and the assumption (ii) of using the dominated convergence theorem. Then (i) and (ii) are clear, for $\{F_{i*} f, F_{i*} g\} = \{f, g\} \circ F_i$ and $F_{i*} X_f = X_{f \cdot F_i}$. □

Thus, if $X_H \in \mathscr{X}(M)'$, ($H \in \mathscr{F}(M)'$) has flow F_t, then F_t is symplectic for each $t \in \boldsymbol{R}$.

Similar limiting statements hold for canonical transformations, provided $F_{i*} \underline{t}$ converges (*cf.* ABRAHAM [1, § 21] for notations). This holds in particular for F_t^i as its generating function is $-H_i$. More precisely, this motivates:

8.8. Definition. *Let (M, ω) be a symplectic manifold and F: $\boldsymbol{R} \times M \to \boldsymbol{R} \times M$ a bijection (measurable). Then F is a* **canonical transformation** *iff there exist canonical transformations F_i: $\boldsymbol{R} \times M \to \boldsymbol{R} \times M$ which are diffeomorphisms with generating functions K_{F_i} such that $F_i \to F$ a.e. and for each $t \in \boldsymbol{R}$, $F_{it} \to F_t$ as in the above* (8.6) *and*

$$K_{F_i} \to K_F \quad in \quad \mathscr{F}(\boldsymbol{R} \times M)'.$$

Thus all the usual equivalences will hold as limiting statements. In particular, the flow F_t of X_H is canonical.

Appendix B: Energy Surfaces

For a smooth Hamiltonian system it is often useful to restrict one's attention to a given energy surface; that is, $H^{-1}(e)$ for $e \in \boldsymbol{R}$ a regular value of H. Then there is induced on $H^{-1}(e)$ a measure preserving flow. (Just what this measure is may be seen from the Hamiltonian flow box theorem: ABRAHAM [1, p. 142].)

One can also consider energy surfaces in the generalized context and a similar result holds as follows:

8.9. Theorem. *Let (M, ω) be a symplectic manifold and X_H a regular Hamiltonian system as in 8.5, with $H_i \to H$. Suppose $e \in \boldsymbol{R}$ is a regular value of each H_i (or a subsequence) and of $H | M \setminus (\operatorname{sing\,supp} H)$ and the flow F_t induces one defined a.e. on the energy surface:*

$$\Sigma_e = \{m \notin \operatorname{sing\,supp} H : H(m) = e\} .$$

This will happen for almost all $e \in \boldsymbol{R}$. Then there exists a smooth measure on Σ_e invariant under the flow induced on Σ_e.

Proof. The first part is clear. For the second, each $m \in \Sigma_e$ lies in a neighborhood of $H_i^{-1}(e)$ for some i, so a measure is inherited on Σ_e. Let $A \subset \Sigma_e$ be measurable and $t \in R$. We must show $\mu(A) = \mu(F_t A)$ where μ is the measure on Σ_e. First note that $\mu(F_t A) = \lim_{l \to \infty} \mu(F_t^l A)$ since $C_{F_t^l A} \to C_{F_t A}$ almost everywhere on Σ_e. (We may assume the topological boundary of $F_t A$, has measure zero.)

However, we also have $\mu(F_t^l(A)) \leqq \mu_l(F_t^l(A))$ where μ_l is the measure on $H_l^{-1}(e)$, since $\mu_l = \mu$ if $F_t^l(A)$ meets Σ_e. Also,

$$\mu_l(F_t^l A) = \mu_l(A) \leqq \mu(A)$$

since μ_l is measure preserving and $A \subset \Sigma_e$. Thus we have

$$\mu(F_t A) \leqq \mu(A).$$

Similarly, using $-t$ for t, we have

$$\mu(a) \leqq \mu(F_t A),$$

giving the result. □

This proof uses critically the fact that the energy surfaces $H_l^{-1}(e)$ actually coincide with Σ_e except on $U_l \downarrow \operatorname{sing\,supp} H$. The more general situation appears to be more delicate, and perhaps the assertion is false there. Notice that the portion of the energy surface belonging to the singular part of H is automatically washed out (has measure zero, corresponding to particles moving infinitly fast along it). In this regard it is instructive to study the examples mentioned in § 6 and in the Introduction.

§ 9. Symmetry Groups and Conservation Laws

In this section we examine the classical method for obtaining conserved quantities, based on symmetry groups, in the generalized setting. This procedure also works in the infinite dimensional case; see MARSDEN [1].

One of the basic ingredients of the theory is the **momentum of a vectorfield**, which turns out to be the conserved quantity. The first main result deals with these momenta. It is a rigorization of SEGAL [1, p. 475] with an adaption from STERNBERG [1, p. 147]. (For basic definitions about Lie groups, see ABRAHAM [1, § 22] or TONDEUR [1].)

9.1. Theorem *(Correspondance Principle). Let $\Phi: G \times M \to M$; $\Phi_g(m) = \Phi(g, m)$ be an action (transformation group) of a Lie group G on the manifold M, which is smooth. Let $\Phi^*: G \times T^* M \to T^* M$ be the corresponding action induced on $T^* M$ given by*

$$\Phi_g^*(\alpha_m) = \alpha_m \circ [T_m \Phi_g]^{-1},$$

where $\alpha_m \in T_m^* M$.

If $X \in \mathscr{X}(M)$ is an infinitesimal generator of Φ and X^* the corresponding one for Φ^*, then

$$X^* = X_{P(X)}$$

(Hamiltonian vectorfield, using the natural symplectic structure on $T^* M$), where

$$P(X) \in \mathscr{F}(T^* M); \quad P(X)(\alpha_m) = \alpha_m(X(m)),$$

and is called the **momentum** *of X.*

Further, for $X, Y \in \mathfrak{X}(M)$, $f \in \mathscr{F}(M)$ we have

$$-P([X, Y]) = \{P(X), P(Y)\}$$

and

$$-(L_X f)^* = \{P(X), f^*\}$$

where $f^* = f \circ \tau^*$, $\tau^*: T^*M \to M$ being the canonical projection.

Remark. The action Φ^* is called **Hamiltonian,** since each infinitesimal generator is globally Hamiltonian. Note that Φ_g^* is symplectic for each $g \in G$. The theorem is also of historical interest in the development of quantum mechanics.

Proof of 9.1. From the chain rule, we have

$$\begin{aligned} \Phi_{gh}^*(\alpha_m) &= \alpha_m \circ [T_m(\Phi_g \circ \Phi_h)]^{-1}, \\ &= \alpha_m \circ [T_n \Phi_g \circ T_m \Phi_h]^{-1}, \qquad n = \Phi_h(m), \\ &= \Phi_g^* \circ \Phi_h^*(\alpha_m), \end{aligned}$$

and from the local formula, Φ^* is smooth, so is an action.

Let F_t be the flow of X, so that F_t^* is the flow of X^*, where

$$F_t^*(\alpha_m) = \alpha_m \circ [T_m F_t]^{-1}.$$

Then we have

$$(F_t^*)_* \theta = \theta$$

where θ is the canonical one-form on T^*M, given by

$$(\theta(\alpha_m) \cdot w_{\alpha_m} = -\alpha_m \circ T\tau^*(w_{\alpha_m})).$$

For this, see ABRAHAM [1, 14.16]. Therefore, $L_{X^*}\theta = 0$, or

$$i_{X^*} d\theta = -d\, i_{X^*}\theta, \quad \text{or} \quad X^* = X_{-\theta(X^*)}.$$

However, we have $T\tau^* \circ X^* = X \circ \tau^*$ since $\tau^* \circ F_t^* = F_t \circ \tau^*$ which means

$$\theta(X^*) \cdot (\alpha_m) = -\alpha_m \cdot T\tau^* \circ X^*(\alpha_m) = P(X) \cdot \alpha_m.$$

In a natural chart, we have

$$P(X) = \sum p_i X^i$$

and with this, a simple computation shows $-P[X, Y] = \{P(X_1), P(X_2)\}$.

For the second formula, note that $L_{X^*} f^* = (L_X f)^*$, from $\tau^* \circ F_t^* = F_t \circ \tau^*$, so that $\{P, f^*\} = -L_{X_{P(X)}} f^* = -L_{X^*} f^*$. The formula may also be proven locally. □

One can define a generalized action of a group on M in much the same way as we did for the special case of flows in 6.1, where now the infinitesimal generators will be generalized vectorfields. In this case all of the above theorem carries over (for $P([X, Y])$, one of X or Y must be smooth) and can in fact be used to obtain non-smooth constants of the motion for smooth Hamiltonian systems (as in 9.3 below). Since this seems to have no important applications and is straightforward anyway, we omit the details.

9.2. Theorem. *Let (M, ω) be a symplectic manifold and Φ a Hamiltonian action of a Lie group on M. Suppose $H \in \mathscr{F}(M)'$ is Hamiltonian regular* (8.2). *$H_t \to H$ and each H_t is invariant; $\Phi_{g*} H_t = H_t$. Then if X_K is an infinitesimal generator of Φ, and F_t is the flow of X_H, we have*

$$K \circ F_t = K \qquad a.e.\ for\ each\ t \in \boldsymbol{R}.$$

Proof. Let G_t be the flow of X_K so that $G_{t*} H_t = H_t$, or $L_{X_K} H_t = 0$, or $L_{X_{H_t}} K = 0$ for all t. Now 6.4 applies. □

If H is invariant under Φ, conditions in the appendix are given (*e.g.*, G is compact) which guarantee the existence of invariant H_t, making H Hamiltonian regular.

In applications, the following special case is the most important.

9.3. Theorem. *Let Φ be an action on M and Φ^* the corresponding Hamiltonian action on $T^* M$. Let $H \in \mathscr{F}(M)'$ be Hamiltonian regular* (8.2) *with $H_t \to H$ and each H_t invariant under Φ^*. Then if X is an infinitesimal generator of Φ, $P(X) \circ F_t = P(X)$, where F_t is the flow of X_H and $P(X)$ is the momentum of X* (9.1).

Proof. Clear from 9.1 and 9.2. □

For example, let $M = \boldsymbol{R}^3$, $T^* M \approx \boldsymbol{R}^6$ and

$$H(q, p) = \tfrac{1}{2} p^2 + \delta_S(q),$$

where S is the unit sphere in $\boldsymbol{R}^3$, and δ_S is the delta function on S.

Then H is Hamiltonian regular with $H_t \to H$ as in 8.5 and H_t invariant under rotations. (This is clear here but is proven generally in the Appendix.) Therefore, by 9.3, angular momentum is conserved under the flow. Physically, the flow corresponds to a particle reflecting elastically from a sphere.

As another example, let g be a pseudo-Riemannian metric on M, smooth or not (see appendix to § 10). We may regard $\tfrac{1}{2} g \in \mathscr{F}(T^* M)'$ and its flow as a Hamiltonian is called the *geodesic flow*. The curve parameter in this case is called proper time. If g is invariant under an action we get conserved quantities by 9.3 (the Lorentz action for example), and integral curves are preserved by the action (Lorentz invariance).

Appendix: Distributions Invariant under an Action

Suppose H is a generalized function on a manifold M and H is invariant under an action Φ of a Lie group G on M. As we saw in the conservation theorems, it is natural to work with smooth functions $H_t \to H$ so H_t are also invariant. Here we show this is possible under simple hypotheses on the action Φ.

9.4. Definition. *Let Φ be a (smooth) action of a Lie group G on a manifold M. We say that an orientable submanifold N of M is a global* **cross-section** *for Φ iff M is the disjoint union of the submanifolds $\Phi_g(N)$ for $g \in G$; that is, M is diffeomorphic to $N \times G$ by Φ.*

For example, the translation group on $\boldsymbol{R}^n$ has a cross section, but the rotation group (compact) does not.

The main result is as follows.

9.5. Theorem. *Let Φ be an action of a Lie group G on a manifold M and H a generalized function invariant under Φ ($\Phi_{g*}H=H$ for all $g\in G$). Suppose either*

(i) *G is compact, or*

(ii) *Φ possesses a cross-section.*

Then there exist smooth functions H_i invariant under Φ and $H_i\to H$. Further, the H_i can be chosen so that 8.5 *holds.*

Proof. (i) Let μ denote (left) Haar measure on G (induced by translates of an orientation at the identity). Suppose $H_i'\in\mathscr{F}(M)$ and $H_i'\to H$. Define a map $H_i\colon M\to \boldsymbol{R}$ by

$$H_i(m)=\int_G H_i'(\Phi_g(m))\,d\mu(g)/\mu(G)$$

for $m\in M$. Clearly H_i is invariant under Φ_g by the change of variables formula. Also, H_i is smooth, since it is the composite of H_i' and Φ_g and we may differentiate under the integral sign. (In fact, it is easy to see that

$$dH_i=\int_G \Phi_{g*}(dH_i')\,d\mu(g)/\mu(G)$$

in the sense that for each $X\in\mathscr{X}(M)$,

$$dH(X)(m)=\int_G [\Phi_{g*}\,dH'(X)]\,(m)\,d\mu(g)/\mu(G).)$$

Finally, we must show that $H_i\to H$. Let $\omega\in\Omega_c^n(M)$ so that $\int H_i'\omega\to H(\omega)$. Then

$$\int_M H_i\omega=\int_M\int_G H_i'\circ\Phi_g(m)\,d\mu(g)\,d\nu(m)/\mu(G)$$

where ν is the (signed) measure of ω. By FUBINI's theorem, this equals

$$\int_G\Big(\int_M H_i'\circ\Phi_g(m)\,d\nu(m)\Big)\,d\mu(g)/\mu(G).$$

Letting $i\to\infty$, we have the result, since $\Phi_{g*}H_i'(\omega)\to\Phi_{g*}H(\omega)=H(\omega)$ and the convergence may be assumed uniform by DE RHAM-SCHWARTZ regularization ($\{\Phi_{g*}\omega\colon g\in G\}$ is a bounded set), see 3.5.

For (ii), define a generalized function H_0 on N by

$$H_0(\alpha)=H(\alpha\wedge\beta)/\int_G\beta$$

for $\alpha\in\Omega_c^k(N)$, $\dim N=k$ and $\beta\in\Omega_c^{n-k}(G)$. This does not depend on β, since $\beta\mapsto H(\alpha\wedge\beta)/\int_G\beta$ is represented by a constant function by invariance of H. Now let $H_{i0}'\to H_0$ and define H_i by $H_i(n,g)=H_{i0}'(n)$. The H_i are smooth, invariant under Φ and $H_i\to H$.

It is easy to argue that we can simultaneously obtain the conditions of 8.5 by a slight modification of the above H_i, in both cases. In fact, find $U_i\downarrow\operatorname{sing\,supp}H$ and h_i such that $h_i=1$ on a neighborhood of $\operatorname{sing\,supp}H$ and with support in U_i (open). Symmetrizing as above (in each case), we may assume h_i are invariant under Φ. Let the new H_i be defined by

$$h_iH_i+(1-h_i)H,$$

which fulfill the requirements. □

If the hypotheses of 9.5 are dropped, it is easy to construct counter examples. For example, consider an action of R on R^2 with a saddle point at (0, 0). Then the delta function at (0, 0) is invariant but cannot be approximated by invariant functions.

§ 10. Applications

This section contains a variety of theorems of more direct physical interest. In 10.1 we prove the classical virial theorem in the smooth, but global case. This is extended in 10.3 to the generalized case. We establish an elementary proposition on mixing in 10.4 and 10.5 proves that the generalized eigenfunctions of a smooth flow uniquely determine it. In the appendix we briefly consider (non-smooth) geodesic flows from the Hamiltonian point of view.

10.1. Theorem (Virial Theorem). *Let M be a manifold with pseudo-Riemannian metric g, and suppose $H=T+V\in\mathscr{F}(T^*M)$, where*

$$T(\alpha_m)=\tfrac{1}{2}\, g(m)\cdot(\alpha_m, \alpha_m)$$

(kinetic energy), and $V\in\mathscr{F}(M)$ (potential energy). Suppose $e\in R$ is a regular value of H and $\Sigma_e=H^{-1}(e)$ is compact (Σ_e may also be a component). For each $X\in\mathfrak{X}(M)$, define the **virial function** *$G_X\in\mathscr{F}(T^*M)$ by*

$$G_X(\alpha_m)=dT(\alpha_m)\cdot X_{P(X)}(\alpha_m)+dV(m)\cdot X(m)$$

where $P(X)$ is the momentum of X (9.1).

Then if F_t is the flow of X_H, we have

(i) $\lim\limits_{t\to\infty}\int\limits_0^t G_X\circ F_s(\alpha_m)\,ds/t=0;$

(ii) $\int G_X\,d\mu_e=0,$

where μ_e is the invariant measure on Σ_e.

Recall that a measure preserving flow is **ergodic** iff whenever a set A is invariant under the flow, A or its complement is of measure zero. Then for each f integrable, we would have

$$\lim_{t\to\infty}\int_0^t f\circ F_s\,ds/t=\int f\,d\mu_e/\mu_e(\Sigma_e)$$

(a constant) almost everywhere, by the Birkhoff ergodic theorem (see HALMOS [3]).

In local coordinates, the function G_X is given by

$$G_X(q^i, p_j)=-g^{ij}(q)\,p_i\,p_k\,\frac{\partial X^k}{\partial q^j}(q)+\frac{1}{2}\,\frac{\partial g^{ij}}{\partial q^k}(q)\,p_i\,p_j\,X^k(q)+\frac{\partial V}{\partial q^i}\,X^i(q)$$

(summation convention); we omit the simple computation. Classically, the second term is omitted; that is, the space is assumed flat.

For the proof of 10.1, we prepare:

10.2. Lemma. *Under the conditions of* 10.1 *(or on any symplectic manifold), if $f\in\mathscr{F}(T^*M)$,*

(i) $\lim\limits_{t\to\infty}\int\limits_0^t \{f,H\}\circ F_s\,ds/t=0$ *on* Σ_e,

and

(ii) $\int\{f,H\}\,d\mu_e=0.$

Proof. On Σ_e,

$$\{f, H\} \circ F_t = \frac{d}{dt}(f \circ F_t),$$

so that as f is bounded, (i) is clear.

To prove (ii), let Ω_e denote the volume on Σ_e. Then on Σ_e,

$$\{f, H\}\, \Omega_e = L_{X_H}(f\,\Omega_e) = d(i_{X_H} f\,\Omega_e).$$

Therefore (ii) follows by 3.2 (iii). □

Proof of 10.1. We claim that $G_X = \{H, P(X)\}$ which, in view of the lemma will give the result. Now

$$\begin{aligned}\{H, P(X)\} &= L_{X_{P(X)}}\, T + L_{X_{P(X)}}\, V \\ &= dT \cdot X_{P(X)}, \quad dV \cdot X_{P(X)}.\end{aligned}$$

However, $dV \cdot X_{P(X)} = dV \cdot X$ (see 9.1). □

In the case of a Hamiltonian regular (8.2) system we proceed as follows:

10.3. Theorem (Generalized Virial Theorem). *Consider a Hamiltonian regular system of the type in* 8.5 *on* $T^* M$ *with each* H_i *of the type in* 10.1; $H_i = T + V_i$. *Suppose all* $H_i^{-1}(e)$ *lie in some compact set and* Ω_e^i *is the volume on* $H_i^{-1}(e)$. *Suppose that* $(dV_i \cdot X)\, \Omega_e^i$ *converges in* $\Omega^n(M)'$ *to*

$$(dV \cdot X)\, \Omega_e + \Omega_s,$$

the first term denoting the smooth portion on Σ_e *and* Ω_s *the "singular part".*

Then if the flow is ergodic on Σ_e, *we have:*

$$\operatorname*{limit}_{t\to\infty} \left(\int_0^t dT \cdot X_{P(X)}\, dt\right) \Big/ t = \left(\int dV \cdot \Omega_e + \int \Omega_s\right) / \mu_e(\Sigma_e).$$

Proof. For each i, we have

$$\int dT \cdot X_{P(X)}\, \Omega_e^i + \int dV_i + \Omega_e^i = 0.$$

Now if we let $i \to \infty$, since $\Omega_e^i \to \Omega_e$ (see 8.9) as measure, the first term converges to

$$\int dT \cdot X_{P(X)}\, \Omega_e,$$

which by ergodicity equals the time average. The second term converges to the right side by 2.8. □

For example, suppose on R^{6n} that

$$H(q, p) = \sum p_i^2/2m_i + \sum V_i(q^i) + \sum_{j<k} V_{jk}(q^j - q^k).$$

Then if we put

$$3n\, T/2 = \operatorname*{limit}_{t\to\infty} \int_0^t \sum p_i^2/2m_i \circ F_s\, ds/t$$

we have (using $X(q)=q$ in 10.3),

$$3n\,T=\operatorname*{limit}_{k\to\infty}\{\int \nabla V_j{}^k(q_j)\cdot q_j\,\Omega_e^l+\sum_{i<j}\nabla V_{ij}^k(q_i-q_j)\cdot(q_i-q_j)\,\Omega_e{}^l\}/\mu_e(\Sigma_e)\,,$$

called the **virial equation of state.** (The right side contains singular terms and smooth ones; for example, the pressure exerted by a wall.)

In summary, the usual formalism of statistical mechanics goes through, but in detail there are non-trivial technicalities. It would be interesting to see the virial equation of state worked out rigorously (*i.e.*, the above limit evaluated) in the case of hard spheres in a box. (It is non-trivial that the flow on Σ_e is ergodic, but otherwise, the hypotheses of 10.3 seem to hold.)

Other elementary theorems of statistical mechanics also hold, such as **equipartition of energy,** in this context. See MARSDEN-WIGHTMAN [4, § 6.2] for details.

Next we prove a theorem on ergodicity. This *may* provide an alternative approach to SINAI's theorem. (See ARNOLD-AVEZ [1, p. 64].)

Recall that a measure preserving flow F_t on a finite measure space M is **mixing** iff for each $A, B\subset M$ measurable,

$$\operatorname*{limit}_{t\to\infty}\mu(F_{-t}(A)\cap B)=\mu(A)\,\mu(B)/\mu(M)\,.$$

(It is enough to verify this for a family generating the measurable sets.) Obviously mixing implies ergodicity. A family of flows is **uniformly mixing** iff the above limit is uniform in F_t.

10.4. Proposition. *Consider a Hamiltonian regular system of the type in* 8.5 *and suppose the smooth flows on the energy surfaces* $H_l^{-1}(e)$, *or components, of* 8.9 *are uniformly mixing for* $A, B\subset\Sigma_e$. *Then the limit flow is mixing on* Σ_e.

Proof. (See proof of 8.9.) We have

$$\operatorname*{limit}_{l\to\infty}\operatorname*{limit}_{t\to\infty}\mu_l(F_{-t}^l(A)\cap B)=\mu(A)\,\mu(B)/\mu(\Sigma_e)$$

and

$$\operatorname*{limit}_{l\to\infty}\mu_l(F_{-t}^l(A)\cap B)=\mu(F_{-t}(A)\cap B)$$

for $A, B\subset\Sigma_e$. By uniformity we may interchange the limits. □

With regard to SINAI's theorem and this technique, there seems to be some hope for getting bounds on the rate of convergence of the limit in the definition of mixing, by using properties of geodesic flows, perhaps in terms of the total curvature. We also propose a similar method for the case of k-systems. It would be interesting to have this program carried out, or even to have these conjectures refuted.

We next give an application to smooth Hamiltonian systems. There are, incidently, several trivial but useful facts. For example, if $m\in M$ and $\Delta_m\in\mathscr{F}(M)'$ is defined by $\delta_m=\Delta_m\,\Omega_\omega$ then m is a critical point of X_H for $H\in\mathscr{F}(M)$ iff $\{H,\Delta_m\}=0$, *cf.* 3.8.

The main theorem is:

10.5. Theorem. *Suppose* $H\in\mathscr{F}(M)$, *where* (M,ω) *is a symplectic manifold, and suppose that the flow of* X_H *is complete. Then the generalized eigenfunctions of* *the flow (or equivalently of iX_H) completely determines the flow. That is, two smooth Hamiltonian flows with the same eigenfunctions are equal.*

To prepare the proof, we recall a few facts from the Gelfand spectral theory (GELFAND-VILENKIN [4]) appropriate to this case.

If X_H has a complete flow F_t, then as we have seen, iX_H is symmetric (and has a self-adjoint extension by STONE's theorem in $L^2(M)$). Then we say $f \in \mathscr{F}(M)'$ is an **eigenfunction** (*complex valued* now!) with eigenvalue λ iff $iL_{X_H} f = \lambda f$ (L_{X_H} given by 3.3 with complexification). From the flow theorem (smoothness essential here), 3.7, this is equivalent to $F_{t*} f = \exp(i\lambda t) f$. For $\lambda \in \mathbf{R}$, let $E_\lambda = \{f \in \mathscr{F}(M)': iX_H(f) = \lambda f\}$ and for $g \in \mathscr{F}_c(M)$, $g_\lambda: E_\lambda \to \mathbf{C}$, $g_\lambda(f) = f(g\,\Omega_\omega)$. The map g_λ is called the **spectral decomposition** of g.

The main spectral theorem is that the real spectrum of iX_H is **complete**; that is $g_\lambda = 0$ for all real λ implies $g = 0$. (This holds for any self-adjoint operator.)

Proof of 10.5. Suppose F_t and G_t are two flows (smooth) with the same eigenfunctions. Let $f \in \mathscr{F}_c(M)$ and $g = F_{t*} f - G_{t*} f$. Since F_t and G_t are diffeomorphisms, $g \in \mathscr{F}_c(M)$ and, by definition $g_\lambda = 0$. Therefore, as the spectrum is complete, $F_{t*} f = G_{t*} f$ for $f \in \mathscr{F}_c(M)$. Therefore by continuity and uniqueness (2.5), $F_{t*} f = G_{t*} f$ for all $f \in \mathscr{F}(M)'$. (Here smoothness is used very strongly again.) Choose $f = \delta_m$ and $F_{t*}\delta_m = \delta_{F_t(m)}$ (3.8) to conclude $\delta_{F_t(m)} = \delta_{G_t(m)}$, proving the assertion. □

Appendix: Generalized Geodesic Flows

Since the motion of a particle in a potential can be thought of as geodesic motion, it is natural to ask what happens to Riemannian geometry when the metric g is not smooth. Here we give a brief indication. See also MARSDEN [5].

10.6. Definition. *A* **generalized pseudo-Riemannian metric** *on a manifold M is a tensor $g \in \mathscr{T}_0^2(M)'$ (contravariant here) which is symmetric and non-degenerate ($g(\alpha, \beta) = 0$ for all $\beta \in \mathscr{X}^*(M)$ implies $\alpha = 0$).*

Let T_g be the kinetic energy of g on T^*M, (locally, $T_g(q,p) = \frac{1}{2} g^{ij}(q)\, p_i p_j$), and suppose the singular support has measure zero. Then the (possibly local) flow determined by 8.5 *for $H = T_g$ is called the* **generalized geodesic flow** *of g, on T^*M.*

Usually g is locally integrable, so that we may relate covariant and contravariant components. From conservation of energy (8.3) we have preservation of the inner product along the flow, wherever that product makes sense.

It seems reasonable to let the metric carry the singular geometric information rather than the differentiable structure of the manifold. This is the point of view we have taken throughout the paper.

Since geodesic motion is a special case of the motion of a Hamiltonian system, all the theorems of § 7–10 apply, so we shall not repeat these here. Instead we discuss connections.

A **generalized connection** is a map $\nabla: \mathscr{X}(M) \times \mathscr{X}(M)' \to \mathscr{X}(M)'$ so that ∇ is $\mathscr{F}(M)$ linear in the first argument, $\mathbf{R}$-linear in the second and $\nabla(X, fY) = f\nabla(X, Y) + Y \cdot (L_X f) \in \mathscr{X}(M)'$ for $X, Y \in \mathscr{X}(M)$, $f \in \mathscr{F}(M)$. As usual we write $\nabla_X Y = \nabla(X, Y)$. Then ∇_X extends as a (generalized) derivation of the full tensor algebra in the usual way. The **torsion** of ∇ is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

In general the curvature

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

is not defined. However if (in local coordinates say) the components of ∇, Γ^i_{jk} are locally bounded, we can define $\frac{1}{2} R^i_{l\,jk}$ as the *generalized* coefficients of

$$d\omega^i_l + \sum_{p=1}^{n} \omega^i_p \wedge \omega^p_l$$

where

$$\omega^i_j = \sum_k \Gamma_{kj}\, dx^k.$$

(See HELGASON [1, p. 44].)

Also, if g is a locally bounded pseudo-Riemannian metric, we can define a corresponding connection ∇ by HELGASON [1, p. 48].

The standard theorems of Riemannian geometry and the calculus of variations break down in the non-smooth case. For example, two arbitrarily close points in the base manifold need not be joined by a geodesic. For a further discussion, see MARSDEN [5].

Note added in proof. (i) The assumption that $H_l = H$ outside U_l in 8.5 [resp. $X_l = X$ in 6.3] may be weakened to: $H_l - H$ is bounded by $\varepsilon_l \to 0$ outside U_l in the C^2 topology [resp. $X_l - X$ is bounded by ε_l in the C^1 topology].

(ii) Some interesting examples of non-uniqueness have been constructed, as well as some theorems which will guarantee uniqueness of flows. See J. MARSDEN, Applied Mathematics Colloquium Lecture (mimeographed), Princeton University, 1968.

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